Kähler geometry

lecture 5

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REMINDER: de Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassmann algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

REMINDER: Cauchy-Riemann equation and Hodge decomposition

The (p,q)-decomposition is defined on differential forms on complex manifold, in a similar way.

DEFINITION: Let (M,I) be a complex manifold A differential form $\eta \in \Lambda^1(M)$ is **of type (1,0)** if $I(\eta) = \sqrt{-1}\,\eta$, and **of type (0,1)** if $I(\eta) = -\sqrt{-1}\,\eta$. The corresponding vector bundles are denoted by $\Lambda^{1,0}(M)$ and $\Lambda^{0,1}(M)$.

REMARK: Cauchy-Riemann equations can be written as $df \in \Lambda^{1,0}(M)$. That is, **a function $f \in C^\infty_{\mathbb{C}}(M)$ is holomorphic if and only if $df \in \Lambda^{1,0}(M)$.**

REMARK: Let (M,I) be a complex manifold, and $z_1, ..., z_n$ holomorphic coordinate system in $U \subset M$, with z_i being holomorphic functions on U. **Then $dz_1, ..., dz_n$ generate the bundle $\Lambda^{1,0}(M)$, and $d\bar{z}_1, ..., d\bar{z}_n$ generate $\Lambda^{0,1}(M)$.**

EXERCISE: Prove this.

REMINDER: The Hodge decomposition on a complex manifold

DEFINITION: Let (M,I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M,I)$ of (p,q)-forms on (M,I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

REMARK: One has $\Lambda^{p,q}(M) = \Lambda^{p,0}(M) \otimes \Lambda^{0,q}(M)$. This gives $\operatorname{rk} \Lambda^{p,q}(M) = \binom{n}{p} \cdot \binom{n}{q}$, where $n = \dim_{\mathbb{C}} M$.

EXERCISE: Prove that the **de Rham differential on a complex manifold has only two Hodge components:**

$$d\,(\Lambda^{p,q}(M)) \subset \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M).$$

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMINDER: Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra**(or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde a \tilde b} ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde a \tilde b}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde a \tilde c}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde L}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

Hodge $*$ operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing:** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n-manifold. Define **the Hodge $*$ operator** $* : \Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge *\beta$.

REMARK: The Hodge $*$ operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, ..., \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k}$, and s the signature of a permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$.

REMARK: $*^2\big|_{\Lambda^k(M)} = (-1)^{k(n-k)} \operatorname{Id}_{\Lambda^k(M)}$

Hodge theory

CLAIM: On a compact Riemannian n-manifold, one has $d^*\Big|_{\Lambda^k M} = (-1)^{nk} * d *$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d(\beta),$$

one has $(d\alpha, *\beta) = (-1)^{\tilde{\alpha}}(\alpha, *d\beta)$. Setting $\gamma := *\beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde{\alpha}}(\alpha, *d(*)^{-1}\gamma) = (-1)^{\tilde{\alpha}}(-1)^{\tilde{\alpha}(\tilde{n}-\tilde{\alpha})}(\alpha, *d*\gamma) = (-1)^{\tilde{\alpha}\tilde{n}}(\alpha, *d*\gamma).$$

■

DEFINITION: The anticommutator $\Delta := \{d, d^*\} = dd^* + d^*d$ is called **the Laplacian** of M. It is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^* (Lemma 1).

THEOREM: (The main theorem of Hodge theory)
There is a basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ.

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

De Rham cohomology

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1} M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$, hence $\alpha \in \ker d \cap \ker d^*$

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^* \alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{=} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} ...$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^* M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4), acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

Reference:

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M. Verbitsky, **Hyperkaehler manifolds with torsion, supersymmetry and Hodge theory,** arXiv:math/0112215, Asian J. Math. Vol. 6, No. 4, pp. 679-712 (2002)

Elena Poletaeva, **Superconformal algebras and Lie superalgebras of the Hodge theory**, arXiv:hep-th/0209168, J.Nonlin.Math.Phys. 10 (2003) 141-147

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = e_i \wedge \eta$. Let $i_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = * e_{v_i} *$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra** $\mathfrak{H}$, with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \operatorname{Id}$.

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: (Lefschetz triples) From the commutator relations in $\mathfrak{H}$, one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k - n$ on k-forms.

COROLLARY: The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra: $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M,I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\,d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\,d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: The Hodge components of d are expressed as $d^{1,0} = \frac{d+\sqrt{-1}\,d^c}{2}$, $d^{0,1} = \frac{d-\sqrt{-1}\,d^c}{2}$. Indeed, $I(\frac{d+\sqrt{-1}\,d^c}{2})I^{-1} = \sqrt{-1}\frac{d+\sqrt{-1}\,d^c}{2}$, hence $\frac{d+\sqrt{-1}\,d^c}{2}$ **has Hodge type (1,0);** the same argument works for $\overline{\partial}$. ■

CLAIM: On a complex manifold, one has $d^c = [\mathcal{W}, d]$.

Proof: Clearly, $[\mathcal{W}, d^{1,0}] = \sqrt{-1}\,d^{1,0}$ and $[\mathcal{W}, d^{0,1}] = -\sqrt{-1}\,d^{0,1}$. Adding these equations, obtain $d^c = [\mathcal{W}, d]$.

COROLLARY: $\{d, d^c\} = \{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1).

De Rham differential on Kaehler manifolds

THEOREM: Let M, I be a complex manifold. Then 1. $\partial^2 = 0$. 2. $\overline{\partial}^2 = 0$. 3. $dd^c = -d^c d$ 4. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

EXERCISE: Prove it.

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

THEOREM: Let M be a Kaehler manifold. One has the following identities **("Kähler idenitities", "Kodaira idenities").**

$$[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [L, \overline{\partial}] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \overline{\partial}^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\overline{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \qquad [L, d^*] = -d^c, \qquad [\Lambda, d^c] = -d^*, \qquad [L, (d^c)^*] = d.$$

Proof: There are two proofs: one uses supersymmetry, for another we prove that a Kähler manifold admits coordinates which are flat up to second order. Neither will be given today. ■

Laplacians and supercommutators

THEOREM: Let

$$\Delta_d := \{d, d^*\}, \quad \Delta_{d^c} := \{d^c, d^{c*}\}, \quad \Delta_\partial := \{\partial, \partial^*\}, \Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\}.$$

Then $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. In particular, **$\Delta_d$ preserves the Hodge decomposition.**

Proof: By Kodaira relations, $\{d, d^c\} = 0$. Graded Jacobi identity gives

$$\{d, d^*\} = -\{d, \{\Lambda, d^c\}\} = \{\{\Lambda, d\}, d^c\} = \{d^c, d^{c*}\}.$$

Same calculation with $\partial, \overline{\partial}$ gives $\Delta_\partial = \Delta_{\overline{\partial}}$.. Also, $\{\partial, \overline{\partial}^*\} = \sqrt{-1}\,\{\partial, \{\Lambda, \partial\}\} = 0$, (Lemma 1), and the same argument implies that **all anticommutators $\partial, \overline{\partial}^*$, etc. all vanish except $\{\partial, \partial^*\}$ and $\{\overline{\partial}, \overline{\partial}^*\}$.** This gives $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$.
■

DEFINITION: The operator $\Delta := \Delta_d$ is called **the Laplacian**.

REMARK: We have proved that **operators $L, \Lambda, d, \mathcal{W}$ generate a Lie superalgebra of dimension $(5|4)$ (5 even, 4 odd), with a 1-dimensional center $\mathbb{R}\Delta$.**

The Lefschetz $\mathfrak{sl}(2)$-action

COROLLARY: The operators L, Λ, H form a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2)$, with relations

$$[L,\Lambda] = H, \quad [H,L] = 2L, \quad [H,\Lambda] = -2\Lambda.$$

DEFINITION: L, Λ, H is called **the Lefschetz $\mathfrak{sl}(2)$-triple**.

REMARK: Finite-dimensional representations of $\mathfrak{sl}(2)$ are semisimple.

REMARK: A simple finite-dimensional representation V of $\mathfrak{sl}(2)$ is generated by $v \in V$ which satisfies $\Lambda(v) = 0$, $H(v) = pv$ **("lowest weight vector")**, where $p \in \mathbb{Z}^{\geqslant 0}$. Then $v, L(v), L^2(v), ..., L^p(v)$ form a basis of $V_p := V$. **This representation is determined uniquely by p.**

REMARK: In this basis, **H acts diagonally:** $H(L^i(v)) = (2i-p)L^i(v)$.

REMARK: One has $V_p = \operatorname{Sym}^p V_1$, where V_1 is a 2-dimensional tautological representation. It is called **a weight p representation of $\mathfrak{sl}(2)$**.

COROLLARY: For a finite-dimensional representation V of $\mathfrak{sl}(2)$, denote by $V^{(i)}$ the eigenspaces of H, with $H\Big|_{V^{(i)}} = i$. **Then L^i induces an isomorphism $V^{(-i)} \xrightarrow{L^i} V^{(i)}$ for any $i > 0$.**

Lefschetz action on cohomology.

From the supersymmetry theorem, the following result follows.

COROLLARY: The $\mathfrak{sl}(2)$-action $\langle L, \Lambda, H\rangle$ and the action of Weil operator commute with Laplacian, hence **preserve the harmonic forms on a Kähler manifold**.

COROLLARY: Any cohomology class can be represented as a sum of closed (p,q)-forms, giving a decomposition $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: odd cohomology of a compact Kähler manifold are even-dimensional.

COROLLARY: Let M be a compact, Kähler manifold of complex dimension n, and $i+p+q=n$. Then L^i defines **the Lefschetz isomorphism** $H^{p,q} \xrightarrow{L^i} H^{p+2i,q+2i}(M)$

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
H^{n,n-3}(M) & & H^{n-1,n-2}(M) & & H^{n-2,n-1}(M) & & H^{n-3,n}(M) \\
\vdots & & \vdots & & \vdots & & \vdots \\
H^{3,0}(M) & & H^{2,1}(M) & & H^{1,2}(M) & & H^{0,3}(M) \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

REMARK: A hyperkähler manifold M is equipped with 3 symplectic forms ω_I, ω_J, ω_K. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic 2-form on (M, I).** ■

THEOREM: (Calabi-Yau) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Supersymmetry in hyperkähler geometry

Let (M, I, J, K, g) be a hyperkaehler manifold, ω_I, ω_J, ω_K its Kaehler forms. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L_I(\alpha) := \omega_I \wedge \alpha$

2. $\Lambda_I(\alpha) := *L_I * \alpha$. It is easily seen that $\Lambda_I = L_J^*$.

3. Three Weil operators $W_I\big|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J\big|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K\big|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}\,(p-q)$

THEOREM: **These operators generate a Lie superalgebra $\mathfrak{a}$** of dimension $(11|8)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence $\mathfrak{a}$ **also acts on the cohomology of M.**

REMARK: The Weil operators form the Lie algebra $\mathfrak{su}(2)$ of unitary quaternions. This means that **the quaternionic action belongs to $\mathfrak{a}$**. In particular, L_J, L_K, Λ_J and Λ_K.

REMARK: The twisted de Rham differentials d_I, d_J, d_K, associated to I, J, K also belong to $\mathfrak{a}$: $d_I = [W_I, d]$, $d_J = [W_J, d]$, $d_K = [W_K, d]$

Supersymmetry and the Hodge decomposition

REMARK: 1. $[L_I, \Lambda_J] = W_K$, $[L_J, \Lambda_K] = W_I$, $[L_I, \Lambda_K] = -W_J$.

2. The even part of $\mathfrak{a}$ **is isomorphic to $\mathfrak{sp}(1,1,\mathbb{H}) \oplus \mathbb{R} \cdot \Delta$.**

3. The odd part $\langle d, d_I, d_J, d_K, d,^* d_I^*, d_J^*, d_K^* \rangle$ **generates the 9-dimensional odd Heisenberg algebra,** with the only non-trivial supercommutators being $\{d, d^*\} = \{d_I, d_I^*\} = \{d_J, d_J^*\} = \{d_K, d_K^*\} = \Delta$

4. The action of $\mathfrak{a}_{even}$ on $\mathfrak{a}_{odd}$ **is the fundamental representation of $\mathfrak{sp}(1,1,\mathbb{H})$ in $\mathbb{H}^2$,** with the quaternionic Hermitian metric on $\mathfrak{a}_{odd}$ provided by the anticommutator.

REMARK: The weight decomposition of the $\mathfrak{sp}(1,1,\mathbb{H}) = \mathfrak{so}(1,4)$-action on $H^*(M)$ **coincides with the Hodge decomposition.**