Kähler geometry

lecture 6

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REMINDER: Holomorphic vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M)\otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f)\otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M)\otimes V \longrightarrow \Lambda^{0,i+1}(M)\otimes V,$$

using $\overline{\partial}(\eta\otimes b) = \overline{\partial}(\eta)\otimes b + (-1)^{\tilde{\eta}}\eta\wedge\overline{\partial}(b)$, where $b\in V$ and $\eta\in\Lambda^{0,i}(M)$.

DEFINITION: A holomorphic vector bundle on a complex manifold (M, I) is a vector bundle equipped with a $\overline{\partial}$-operator which satisfies $\overline{\partial}^2 = 0$. In this case, $\overline{\partial}$ is called **a holomorphic structure operator**.

EXERCISE: Consider the Dolbeault differential $\overline{\partial}: \Lambda^{p,0}(M)\longrightarrow\Lambda^{p,1}(M) = \Lambda^{p,0}(M)\otimes\Lambda^{0,1}(M)$. **Prove that it is a holomorphic structure operator on $\Lambda^{p,0}(M)$.**

DEFINITION: The corresponding holomorphic vector bundle $(\Lambda^{p,0}(M), \overline{\partial})$ is called **the bundle of holomorphic p-forms**, denoted by $\Omega^p(M)$.

REMINDER: Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\bar\partial\; B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\; V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}:\; V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\bar\partial$.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMINDER: Calabi-Yau manifolds

DEFINITION: Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$. The holomorphic line bundle $(\Lambda^{n,0}(M), \overline{\partial})$ is called **canonical bundle** of M.

DEFINITION: A Calabi-Yau manifold is a compact Kaehler manifold with topologically trivial canonical bundle.

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as $(\alpha, \alpha') \longrightarrow \frac{\alpha\wedge\overline{\alpha}'}{\omega^n}$. Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\operatorname{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

REMARK: Its proof uses spinors.

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$.**

DEFINITION: A holomorphically symplectic Ricci-flat Kaehler manifold is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Ricci-flat Riemannian manifold with $\pi_1(M)$ infinite. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

COROLLARY: A fundamental group of a compact Ricci-flat Riemannian manifold is **"virtually polycyclic": it is projected to a free abelian subgroup with finite kernel.**

REMARK: This is equivalent to any compact Ricci-flat manifold having a finite covering which has free abelian fundamental group.

REMARK: This statement contains the Bieberbach's solution of Hilbert's 18-th problem on classification of crystallographic groups.

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}\mathrm{ol}(M_l) = Sp(n_l)$, $\mathcal{H}\mathrm{ol}(K_l) = SU(m_l)$

REMINDER: The Hodge decomposition on cohomology

THEOREM: On a compact Kaehler manifold M, **the Hodge decomposition is compatible with the Laplace operator.** This gives a decomposition of cohomology, $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: $H^p(M)$ **is even-dimensional for odd** p.

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

REMARK: $H^{p,0}(M)$ **is the space of holomorphic** p**-forms.** Indeed, $dd^* + d^*d = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$ hence **a holomorphic form on a compact Kähler manifold is closed.**

Holomorphic Euler characteristic

DEFINITION: A holomorphic Euler characteristic $\chi(M)$ of a Kähler manifold is a sum $\sum(-1)^p \dim H^{p,0}(M)$.

THEOREM: (Riemann-Roch-Hirzebruch) For an n-fold, $\chi(M)$ **can be expressed as a polynomial expressions of the Chern classes,** $\chi(M) = td_n$ where td_n is an n-th component of the Todd polynomial,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

REMARK: The Chern classes are obtained as polynomial expression of the curvature (Gauss-Bonnet). Therefore $\chi(\tilde{M}) = p\chi(M)$ **for any unramified** p**-fold covering** $\tilde{M} \longrightarrow M$**.**

REMARK: Bochner's vanishing and the classical invariants theory imply:

1. When $\mathcal{H}\mathrm{ol}(M) = SU(n)$, we have $\dim H^{p,0}(M) = 1$ **for** $p = 1, n$**, and 0 otherwise.** In this case, $\chi(M) = 2$ for even n and 0 for odd.

2. When $\mathcal{H}\mathrm{ol}(M) = Sp(n)$,we have $\dim H^{p,0}(M) = 1$ **for even** p $0 \leqslant p \leqslant 2n$**, and 0 otherwise.** In this case, $\chi(M) = n + 1$.

COROLLARY: $\pi_1(M) = 0$ if $\mathcal{H}\mathrm{ol}(M) = Sp(n)$, or $\mathcal{H}\mathrm{ol}(M) = SU(2n)$. If $\mathcal{H}\mathrm{ol}(M) = SU(2n+1)$, $\pi_1(M)$ is finite.

REMINDER: Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0} b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0} b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0} b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is, $\Theta_B = -2\partial\bar\partial \log |b|$.**

COROLLARY: If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

$\partial\bar\partial$-lemma

THEOREM: ("$\partial\bar\partial$-lemma")
Let M be a compact Kaehler manifold, and $\eta\Lambda^{p,q}(M)$ an exact form. **Then $\eta = \partial\bar\partial\alpha$, for some $\alpha \in \Lambda^{p-1,q-1}(M)$.**

Proof. Step 1: Write $\eta = \sum_\alpha \eta_\alpha$, where $d\eta_\alpha = 0$ and $\Delta\eta_\alpha = \alpha\eta_\alpha$. **This is possible because of Hodge theory.**

Step 2: Since $[d, \Delta] = 0$, d preserves the eigenspaces of Δ. This implies $d\eta_\alpha = 0$. Then $\eta_\alpha = \frac{1}{\alpha}\Delta\eta_\alpha = \frac{1}{\alpha}dd^*\eta_\alpha$. It would suffice to prove that **each η_α belongs to the image of $\partial\bar\partial = -\frac{\sqrt{-1}}{2}dd^c$, where $d^c = IdI^{-1}$.**

Step 3: We have $d^* = -[\Lambda, d^c$ (Kähler identities). Also, $d^c\eta_\alpha = (\sqrt{-1})^{q-p}Id\eta_\alpha = 0$, hence $d^*(\eta_\alpha) = d^c\Lambda\eta_\alpha$. Using Step 2, this implies

$$\eta_\alpha = \frac{1}{\alpha}dd^*\eta_\alpha = \frac{1}{\alpha}dd^c\Lambda\eta_\alpha.$$

■

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M, ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$. If ω_1 is a new Kaehler metric on (M, I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$.

REMARK: For two metrics ω_1, ω in the same Kähler class, one has $\omega_1 - \omega = dd^c\varphi$, **for some function** φ (dd^c-lemma).

COROLLARY: A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + dd^c\varphi)^n = \omega^n e^f$, where $-2\partial\overline{\partial}f = \Theta_{K,\omega}$ **(such** f **exists by** $\partial\overline{\partial}$**-lemma).**

Proof. Step 1: For such f, φ, one has $\log\frac{h}{h_1} = -\log e^f = -f$. As shown above, the corresponding curvatures are related as $\Theta_{K,\omega_1} - \Theta_{K,\omega} = -2\partial\overline{\partial}\log(h/h_1)$. This gives

$$\Theta_{K,\omega_1} = \Theta_{K,\omega} + -2\partial\overline{\partial}\log(h/h_1) = \Theta_{K,\omega} - 2\partial\overline{\partial}f.$$

Proof. Step 2: Therefore, ω_1 **is Ricci-flat if and only if** $\Theta_{K,\omega} - 2\partial\overline{\partial}f$**.** ■

To find a Ricci-flat metric **it remains to solve an equation** $(\omega + dd^c\varphi)^n = \omega^n e^f$ **for a given** f**.**

The complex Monge-Ampère equation

To find a Ricci-flat metric **it remains to solve an equation $(\omega + dd^c\varphi)^n = \omega^n e^f$ for a given f.**

THEOREM: (Calabi-Yau) Let (M, ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function φ** such that $(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

DEFINITION:

$$(\omega + \sqrt{-1}\,\partial\bar\partial\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By construction, one has $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$. **We need to show** $\psi = const$**.**

Step 2: $\omega_2 = \omega_1 + \sqrt{-1}\,\partial\bar\partial\psi$ gives

$$0 = (\omega_1 + \sqrt{-1}\,\partial\bar\partial\psi)^n - \omega_1^n = \sqrt{-1}\,\partial\bar\partial\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $\sqrt{-1}\,\partial\bar\partial\psi \wedge P = 0$, this gives $\psi\partial\bar\partial\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\bar\partial\psi \wedge P = -\int_M \partial\psi \wedge \bar\partial\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\bar\partial\psi = 0$.** ∎