Kähler geometry

lecture 7

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REMINDER: Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra**(or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

REMINDER: Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\} : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even or odd element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + (-1)^{\tilde{L}}\{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

REMINDER: Hodge $*$ operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing:** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n-manifold. Define **the Hodge $*$ operator** $* : \Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge *\beta$.

REMARK: The Hodge $*$ operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, ..., \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k}$, and s the signature of a permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$.

REMARK: $*^2\big|_{\Lambda^k(M)} = (-1)^{k(n-k)} \operatorname{Id}_{\Lambda^k(M)}$

REMINDER: Supersymmetry in Kähler geometry

Let (M,I,g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, $d^* := *d*$, $\Delta := \{d, d^*\}$, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L*\alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4), acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

REMINDER: Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

REMINDER: Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M,I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

THEOREM: A simple hyperkähler manifold is always simply connected.

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification) Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3** is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

Supersymmetry in hyperkähler geometry

Let (M, I, J, K, g) be a hyperkaehler manifold, ω_I, ω_J, ω_K its Kaehler forms. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L_I(\alpha) := \omega_I \wedge \alpha$

2. $\Lambda_I(\alpha) := *L_I * \alpha$. It is easily seen that $\Lambda_I = L_J^*$.

3. Three Weil operators $W_I\big|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J\big|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K\big|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension $(11|8)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence $\mathfrak{a}$ **also acts on the cohomology of M.**

REMARK: The Weil operators form the Lie algebra $\mathfrak{su}(2)$ of unitary quaternions. This means that **the quaternionic action belongs to $\mathfrak{a}$**. In particular, L_J, L_K, Λ_J and Λ_K.

REMARK: The twisted de Rham differentials d_I, d_J, d_K, associated to I, J, K also belong to $\mathfrak{a}$: $d_I = [W_I, d]$, $d_J = [W_J, d]$, $d_K = [W_K, d]$

Supersymmetry and the Hodge decomposition

REMARK: 1. $[L_I, \Lambda_J] = W_K$, $[L_J, \Lambda_K] = W_I$, $[L_I, \Lambda_K] = -W_J$.

2. The even part of $\mathfrak{a}$ **is isomorphic to $\mathfrak{sp}(1,1,\mathbb{H}) \oplus \mathbb{R} \cdot \Delta$.**

3. The odd part $\langle d, d_I, d_J, d_K, d,^* d_I^*, d_J^*, d_K^* \rangle$ **generates the 9-dimensional odd Heisenberg algebra,** with the only non-trivial supercommutators being $\{d, d^*\} = \{d_I, d_I^*\} = \{d_J, d_J^*\} = \{d_K, d_K^*\} = \Delta$

4. The action of $\mathfrak{a}_{even}$ on $\mathfrak{a}_{odd}$ **is the fundamental representation of $\mathfrak{sp}(1,1,\mathbb{H})$ in $\mathbb{H}^2$,** with the quaternionic Hermitian metric on $\mathfrak{a}_{odd}$ provided by the anticommutator.

REMARK: The weight decomposition of the $\mathfrak{sp}(1,1,\mathbb{H}) = \mathfrak{so}(1,4)$-action on $H^*(M)$ **coincides with the Hodge decomposition.**

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** Tw(M) of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let Tw$(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathsf{Tw}} = I_m \oplus I_J : T_x \mathsf{Tw}(M) \to T_x \mathsf{Tw}(M)$ satisfies $I_{\mathsf{Tw}}^= - \operatorname{Id}$. **It defines an almost complex structure on Tw(M).** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, Tw$(M) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, Tw(M) never admits a Kähler structure.

$SU(2)$-action on the cohomology and its applications

OBSERVATION: From the supersymmetry result, we obtain **$SU(2)$-action on cohomology,** containing the $U(1)$-action from the induced complex structures.

DEFINITION: Trianalytic subvarieties are closed subsets which are complex analytic with respect to I, J, K.

REMARK: Let $[Z]$ be a fundamental class of a complex subvariety Z on a Kähler manifold. **Then Z is (1,1)-invariant.**

COROLLARY: A fundamental class of a trianalytic subvariety is $SU(2)$-invariant.

THEOREM: Let M be a hyperkähler manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$, such that for any induced complex structure $L \in \mathbb{C}P^1 \backslash S$, **all compact complex subvarieties of (M, L) are trianalytic.**

Its proof is based on **Wirtinger's inequality**.

Wirtinger's inequality

PROPOSITION: (Wirtinger's inequality)
Let $V \subset W$ be a real $2d$-dimensional subspace in a complex Hermitian vector space (W, I, g), and ω its Hermitian form. **Then $\operatorname{Vol}_g V \geqslant \frac{1}{2^d d!} \omega^d|_V$, and the equality is reached only if V is a complex subspace.**

COROLLARY: Let (M, I, ω, g) be a Kähler manifold, and $Z \subset M$ its real subvariety of dimension $2d$. Then $\int_Z \operatorname{Vol}_Z \geqslant \frac{1}{2^d d!} \int_Z \omega^d$, **and the equality is reached only if Z is a complex subvariety.**

REMARK: Notice that $\int_Z \omega^d$ is a (co)homology invariant of Z, and stays constant if we deform Z. Therefore, **complex subvarieties minimize the Riemannian volume in its deformation class**.

Wirtinger's inequality for hyperkähler manifolds

DEFINITION: Let (M,I,J,K,g) be a hyperkähler manifold, and $Z\subset M$ a real $2d$-dimensional subvariety. Given an induced complex structure $L=aI+bJ+cK$, define **the degree** $\deg_L(Z):=\frac{1}{2^d d!}\int_Z\omega_L^d$, where $\omega_L(x,y)=g(x,Ly)$, which gives $\omega_L=a\omega_I+b\omega_J+c\omega_K$.

Proposition 1: Let $Z\subset(M,L)$ be a complex analytic subvariety of (M,L). (a) Then $\deg_L(Z)$ **has maximum at** L**.** (b) Moreover, this maximum is absolute and **strict, unless** $\deg_L(Z)$ **is constant as a function of** L**.** (c) In the latter case, Z is trianalytic.

Proof. Step 1: By Wirtinger's inequality, $\operatorname{Vol}_g Z\geqslant\deg_L(Z)$, and the equality is reached if and only if Z is complex analytic in (M,L). This proves (a).

Step 2: If the maximum is not strict, there are two quaternions L and L' such that Z is complex analytic with respect to L and L'. This means that TZ **is preserved by the algebra of quaternions generated by** L **and** L'**,** hence Z **is trianalytic, and** $\deg_L(Z)$ **constant**. This proves (b) and (c). ■

Trianalytic subvarieties in generic induced complex structures

THEOREM: Let M be a hyperkähler manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$, such that for any induced complex structure $L \in \mathbb{C}P^1 \backslash S$, **all compact complex subvarieties of (M, L) are trianalytic.**

Proof. Step 1: Let $R \subset H^2(M, \mathbb{Z})$ be the set of all integer cohomology classes $[Z]$, for which the function $\deg_L([Z]) = \int_{[Z]} \omega_L^d$ is not constant, and S the set of all strict maxima of the function $\deg_L([Z])$ for all $[Z] \in R$. **Then S is countable.** Indeed, $\deg_L([Z])$ is a polynomial function.

Step 2: Now, let $L \in \mathbb{C}P^1 \backslash S$. For all complex subvarieties $Z \subset (M, L)$, $\deg_L([Z])$ cannot have strict maximum in L. By Proposition 1 (c), **this implies that Z is trianalytic.** ■

COROLLARY: For M compact and hyperkähler, and $L \in \mathbb{C}P^1$ generic, **the manifold (M, L) has no complex divisors.** In particular, **it is non-algebraic**