Calabi-Yau theorem for Hessian manifolds

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Estruturas geométricas em variedades

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Flat affine manifolds

DEFINITION: A flat affine manifold is a smooth manifold equipped with a flat, torsion-free connection.

EXAMPLE: Let Γ be a group acting on an open subset $U \subset \mathbb{R}^n$ freely, properly discontinuously, by affine transforms. **Then U/Γ is affine.**
REMARK: A simply connected, flat affine manifold M **admits a map $M \longrightarrow \mathbb{R}^n$ which is compatible with the flat affine structure**.

DEFINITION: A flat affine manifold (M, ∇) is called **special affine** if it admits a ∇-invariant volume form, or, equivalently, it its holonomy belongs to $SL(n, \mathbb{R})$. A flat affine manifold is **complete** if it is a quotient of $\mathbb{R}^n$ by a discrete group of affine transforms.

CONJECTURE: Markus conjecture (1961)
a compact affine manifold is complete if and only if it is special.

CONJECTURE: Auslander conjecture (1964)
For any complete affine manifold $\mathbb{R}^n/\Gamma$, the group Γ is solvable.

CONJECTURE: Chern conjecture (1955)
The Euler class of a compact affine manifold vanishes (proven by Bruno Klingler in 2017 for special affine manifolds).

Hessian manifolds

DEFINITION: Let (M,∇) be a flat affine manifold. Since ∇ is torsion-free, the tensor $\nabla^2(f)$ is symmetric for any $f \in C^\infty M$. A Riemannian metric g on M is called **Hessian** if locally $g = \nabla^2(f)$, for some f which is called **the potential** of the metric.

EXERCISE: Prove that **a metric g on (M,∇) is Hessian if and only if $\nabla(g)$ is a symmetric 3-form.**

EXERCISE: Prove that the complex

$$C^\infty(M) \xrightarrow{\nabla^2} \operatorname{Sym}^2 T^*M \xrightarrow{\nabla} \frac{\Lambda^1(M)\otimes \operatorname{Sym}^2 T^*M}{\operatorname{Sym}^3 T^*M}$$

is elliptic.

REMARK: Cohomology of this complex are called **Hessian cohomology**, and the cohomology classes represented by Hessian metrics are called **Hessian classes**.

REMARK: This geometry **is in many ways similar to Kähler geometry.**

Calabi-Yau theorem

DEFINITION: A cohomology class of a Kähler form in $H^2(M)$ is called **the Kähler class** of a Kähler manifold.

THEOREM: (Calabi-Yau theorem) Let (M, I) be a compact complex manifold. **A Kähler form ω is uniquely, up to a constant multiplier, determined by its Kähler class and its volume form.**

This theorem has a Hessian counterpart.

THEOREM: (S.-Y. Cheng, S.-T. Yau, 1982)
Let (M, ∇) be a compact special affine manifold. **A Hessian metric on M is uniquely, up to a constant multiplier determined by its Hessian class and its volume form.**

COROLLARY: A compact special affine Hessian manifold **is a finite quotient of a torus.**

Special affine manifolds with integral monodromy

DEFINITION: Let (M, ∇) be an affine manifold, and $\pi_1(M) \longrightarrow GL(T_x M)$ its monodromy representation. We say that M **has integral monodromy** if there exists a lattice $\Lambda \subset T_x M$ preserved by the action of $\pi_1(M)$.

PROPOSITION: If (M, ∇) is an oriented flat affine manifold with integral monodromy, **it is special affine.**

Proof: Let $GL^+(n, \mathbb{R})$ denote the connected component. Then $GL^+(n, \mathbb{R}) \cap GL(n, \mathbb{Z}) \subset SL(n, \mathbb{Z})$ because **the determinant of any $A \in GL(n, \mathbb{Z})$ is integral and invertible,** hence $\det A = \pm 1$. ■

THEOREM: Let M be a compact, special, Hessian flat affine manifold with integral monodromy. Then **a Hessian metric on M is uniquely, up to a constant multiplier, determined by its Hessian class and its volume form.**

THEOREM: Let M be a compact, special, Hessian flat affine manifold with integral monodromy. **Then M is a finite quotient of a torus.**

REMARK: In this generality, the result **can be deduced from the Calabi-Yau theorem.** Without the integral monodromy assumption, it is also true, but **one needs to reproduce the steps of the proof of Calabi-Yau theorem.**

Ehresmann connections

REMARK: Let (B, ∇) be a vector bundle with connection on a manifold M. Then the total space $\operatorname{Tot}(B)$ admits a direct sum decomposition $T \operatorname{Tot}(B) = T_{\text{ver}} \operatorname{Tot}(B) \oplus T_{\text{hor}} \operatorname{Tot}(B)$, where $T_{\text{ver}} \operatorname{Tot}(B)$ is **the fiberwise tangent space** of all vectors tangent to the fibers of the projection $TB \longrightarrow M$, and $T_{\text{hor}} \operatorname{Tot}(B)$, called **the horizontal tangent space** is generated by the tangent vectors to the solutions of $\nabla_{\gamma'(t)} b(t) = 0$, where $\gamma: [0,1] \longrightarrow M$ is a path, and $b(t)$ a section of $B|_\gamma$.

CLAIM: A connection ∇ **is uniquely determined by the decomposition $T \operatorname{Tot}(B) = T_{\text{ver}} \operatorname{Tot}(B) \oplus T_{\text{hor}} \operatorname{Tot}(B)$.**

REMARK: A decomposition $T \operatorname{Tot}(B) = T_{\text{ver}} \operatorname{Tot}(B) \oplus T_{\text{hor}} \operatorname{Tot}(B)$ is called **an Ehresmann connection**.

Ehresmann connections and an almost complex structure

CLAIM: Let $T\operatorname{Tot}(TM) = T_{\mathsf{ver}}\operatorname{Tot}(TM) \oplus T_{\mathsf{hor}}\operatorname{Tot}(TM)$ be am Ehresmann connection on TM. Then, **for each $v \in \operatorname{Tot}(M)$, one has $T_{\mathsf{ver}}\operatorname{Tot}(M)|_v = T_{\pi(v)}M$ and $T_{\mathsf{hor}}\operatorname{Tot}(M)|_v = T_{\pi(v)}M$.**

PROPOSITION: Let $T\operatorname{Tot}(TM) = T_{\mathsf{ver}}\operatorname{Tot}(TM) \oplus T_{\mathsf{hor}}\operatorname{Tot}(TM)$ be am Ehresmann connection on TM. Define an almost complex structure $I \in \operatorname{End}(\operatorname{Tot}(TM))$ by setting $I(a,b) = (b,-a)$, where $(a,b) \in T_{\mathsf{ver}}\operatorname{Tot}(M)|_v = T_{\pi(v)}M \oplus T_{\pi(v)}$. **Then I is integrable if the corresponding connection on TM is flat and torsion-free.**

Proof: If M is flat and torsion-free, it can be identified with $\mathbb{R}^n$ in such a way that the coordinate vector fields are parallel, and then I maps the parallel coordinate vector fields to parallel coordinate vector fields, hence $\operatorname{Tot}TM$ **admits a coordinate system such that I is constant.** ■

Complex double manifold

CLAIM: Let (M, ∇) be a flat affine manifold with integral monodromy, and $\Lambda_x \in T_x M$ be the integral lattice preserved by the monodromy. Denote by $\Lambda_M \subset \mathrm{Tot}(TM)$ the manifold obtained by taking all $v \in \Lambda_x$ and transporting them to all $T_y M$ using the parallel transport along ∇. Then $\Lambda_M \subset \mathrm{Tot}(TM)$ is a smooth submanifold in TM, and $\Lambda_y = \Lambda_M \cap T_y M$ is a sublattice in $T_y M$. ■

DEFINITION: The quotient $DM := \mathrm{Tot}(TM)/\Lambda_M$ is called **the complex double of M**. The fibers $T_y M/\Lambda_y$ of the natural projection $\pi : TM/\Lambda_M \longrightarrow M$ are compact tori.

REMARK: Since the action of Λ_M on $\mathrm{Tot}\, TM$ commutes with the complex structure on $\mathrm{Tot}\, TM$, **the complex double of M is a complex manifold, compact when M is compact.**

The Kähler potential and the Hessian

DEFINITION: A $(1,1)$-form ω on an almost complex manifold M is called **a Hermitian form** if $\omega(Ix,x) > 0$ for each non-zero tangent vector $x \in T_m M$. The corresponding Riemannian form $d(x,x) := \omega(Ix,x)$ is called **a Hermitian metric**.

DEFINITION: Let (M,I) be a complex manifold, and $d^c := I^{-1} dI : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$ be **the twisted differential**. The map $f \longrightarrow dd^c f$ taking a function f to $dd^c f \in \Lambda^{1,1}(M)$ is called **the pluri-Laplacian**. A Hermitian form which can be locally represented as $dd^c f$ is called **Kähler** and f its **Kähler potential**. In this situation f is called **strictly plurisubharmonic**.

REMARK: Let ∇ be a flat, torsion-free connection on M, and let $\mathrm{Alt} : T^*M \otimes T^*M \longrightarrow \Lambda^2(M)$ be the antisymmetrizator. **Then** $dd^c f = \mathrm{Alt}(\nabla I(df)) = \mathrm{Alt}(I \otimes \mathrm{Id}(\mathrm{Hess}(f))$, where $\mathrm{Hess}(f) = \nabla(df) \in T^*M \otimes T^*M$.

COROLLARY: Consider a flat affine manifold equipped with a flat connection preserving a complex structure I. Then $dd^c f(Ix,x) = \frac{1}{2}(\mathrm{Hess}(f)(x,x) + \mathrm{Hess}(f)(Ix,Ix)$. In particular, **any convex function gives a Kähler potential.**

The Kähler metrics on $\operatorname{Tot}(TM)$

CLAIM 1: Let ∇ be a flat, torsion-free connection on M, and $\operatorname{Tot}(TM) \xrightarrow{\pi} M$ the tangent space of M equipped with the standard complex structure. Then **f is strictly convex if and only if $\pi^* f$ is strictly plurisubharmonic**.

Proof: $dd^c f(Ix, x) = \frac{1}{2}(\operatorname{Hess}(f)(x,x) + \operatorname{Hess}(f)(Ix, Ix))$. If x is horizontal, $I(x)$ is vertical and vice versa, hence it suffices to check the positivity only when x is horizontal. In this case $\operatorname{Hess}(f)(Ix, Ix)$ is identically zero, hence $dd^c f(Ix, x) > 0$ if and only if $\operatorname{Hess}(f)(x, x) > 0$. ■

The Kähler metrics on $\operatorname{Tot}(TM)$ (2)

PROPOSITION: Consider a Hessian metric g on M, and let $\pi^* g$ be its pullback to $\operatorname{Tot}(TM)$. **Then $\pi^* g + I(\pi^* g)$ is a Kähler metric on $\operatorname{Tot}(TM)$.** Conversely, **if h is a Kähler metric on $\operatorname{Tot}(TM)$ which is invariant under the parallel translation along the fibers of π can be obtained this way.**

Proof. Step 1: The first statement immediately follows from Claim 1. To prove the converse, we use the complex double. Since the statement is local in M, we can always assume that the monodromy of M is trivial, hence the complex double exists. Consider $x, y \in T_m M$, and let $g(x,y) := h(x_{\text{hor}}, y_{\text{hor}})$, where $x_{\text{hor}}, y_{\text{hor}}$ are horizontal lifts of x, y. Then $h(a,b) = g(a,b) + g(Ia, Ib)$ because h is I-invariant. It remains to show that g is Hessian.

Step 2: To show that g is Hessian, consider a Kähler potential for h. Locally, it is well defined. Let $U \subset M$ be a sufficiently small open set. The obstruction to the existence of the Kähler potential for h is $H^1(DU, \ker dd^c) = H^1(DU, \operatorname{Re} \mathcal{O})$: the sheaf $\ker dd^c$ coincides with the shea of real parts of holomorphic functions. This group vanishes, because $\pi^{-1}(U)$ is Stein when $U \subset \mathbb{R}^n$ is convex, hence $H^1(\pi^* \pi, \ker dd^c) = 0$, and h admits a global Kähler potential φ on DU. Averaging φ with the torus action, we obtain another Kähler potential φ_0, which is constant on the fibers of π. This implies that $\pi^* \varphi_0$ is strictly plurisubharmonic, hence φ_0 is convex by Claim 1, and $h = \pi^* g + I(\pi^* g)$, also by Claim 1. ■

Complex Monge-Ampère Equation

DEFINITION: Let φ be a function on $\mathbb{C}^n$, and $dd^c\varphi$ its **complex Hessian**, $dd^c\varphi := \operatorname{Hess}(\varphi) + I(\operatorname{Hess}(\varphi))$. It is a Hermitian form.

DEFINITION: Let (M, g) be a Kaehler n-manifold. **The complex Monge-Ampere equation is**

$$(\omega + dd^c\varphi)^n = e^f \omega^n$$

THEOREM: (Calabi-Yau) On a compact Kähler manifold, **the complex Monge-Ampere equation has a unique solution,** for any smooth function f subject to constraint $\int_M e^f \operatorname{Vol}_g = \int_M \operatorname{Vol}_g$.

Let $\omega_1, \omega_2 = \omega_1 + dd^c\varphi$ be two solutions of the same Monge-Ampère equations on a complex manifold **(not necessarily compact)** $\omega_1^n = \omega_2^n$. Then $\varphi \in \ker D$, where D is the differential operator written as $D(u) = \frac{dd^c u \wedge P}{\omega_i^n}$, and $P = \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-i-1}$.

CLAIM: The operator D is elliptic.

COROLLARY: By E. Hopf maximum principle, **φ cannot have a local maximum, unless it is constant.**

The Kähler potential on a complex double is constant on its fibers

COROLLARY 1: Let M be a compact flat affine manifold, and ω a Kähler form obtained from a Hessian metric on its double $DM \xrightarrow{\pi} M$. **Then any solution of the Monge-Ampére equation $(\omega + dd^c\varphi)^n = e^{\pi^* f} f\omega^n$ is constant on the fibers of the projection $\pi : DM \longrightarrow M$.**

Proof: Let $x, y \in \pi^{-1}(m)$ be two points such that $\varphi(x) - \varphi(y)$ is maximal for all x, y and m, and denote by L the parallel shift of the torus which maps y to x. The map L is well defined in a small neighbourhood $U \ni m$ in M (globally, it is not necessarily defined, unless it is monodromy invariant). Then φ and $L^*\varphi$ are two solutions of the same Monge-Ampere, which implies that $\varphi - L^*\varphi \in \ker D$ is maximal in x. **This is impossible by the maximum principle.** ■

Cheng-Yau obtained from the complex Monge-Ampère on the double

Let $\det(u)$ denote the Riemannian volume form of a Riemannian metric u.

THEOREM: (a special case of Cheng-Yau theorem)
Let (M, g, ∇) be a compact Hessian manifold with integral monodromy and Vol a parallel volume form. Then **for any smooth function f subject to constraint $\int_M e^f \operatorname{Vol} = \int_M \operatorname{Vol}$, the equation $\det(g + \operatorname{Hess}\varphi) = e^f \operatorname{Vol}$ has a unique solution.**

Proof: Consider the Hermitian form ω on the double DM associated with g as in Claim 1. Let V be the parallel volume form of DM. Then $\pi^*(e^f)V = (\omega + dd^c\varphi)^n$ has a unique solution by Calabi-Yau theorem. By Corollary 1, φ is constant on the fibers of $DM \xrightarrow{\pi} M$, which gives $\varphi = \pi^*\varphi_0$. By Claim 1, $g + \operatorname{Hess}\varphi_0$ is a Hessian metric which has volume $e^f \operatorname{Vol}$. ∎

COROLLARY: Let (M, g, ∇) be a compact Hessian manifold with integral monodromy and Vol a parallel volume form. **Then there exists a Hessian metric g such that $\det g = \operatorname{Vol}$.** ∎

Complete Hessian metrics with constant volume

THEOREM: Let M be a special Hessian manifold, Vol the constant volume form, and g a Hessian metric on M such that $\mathrm{Vol} = \det(g)$. **Then the universal covering of M is $\mathbb{R}^n$, and g is a constant Riemannian form.**

Proof. Step 1: Let $\tilde{M}$ be the universal covering of M. The parallel 1-forms on $\tilde{M}$ are closed, hence exact, $\theta_i = dx_i$. These functions define an affine map $\tilde{M} \xrightarrow{\delta} \mathbb{R}^n$ which is affine, called **the development map**. By a theorem of Koszul, for any Hessian manifold, δ is injective, and its image is convex.

Step 2: Let κ be the Hessian of the pullback of g on $\tilde{M} \subset \mathbb{R}^n$. The graph of κ is **an affine hypersphere**, in the terminology of Blaschke, and a closed affine hypersphere in $\mathbb{R}^{n+1}$ is a graph of a quadratic polynomial (Calabi-Pogorelov).

Step 3: To prove that the graph of κ is complete, use completeness of the pullback of g to $\tilde{M}$ to obtain that $\int |\kappa'| dt$ is infinite on any affine line in $\tilde{M}$. ■

REMARK: In this situation, the metric g is constant, $\nabla(g) = 0$. In particular, **∇ is the Levi-Civita connection.**

Bieberbach theorem for Hessian manifolds

COROLLARY: Any a compact special Hessian manifold with integral monodromy **is a finite quotient of a torus.**

Proof. Step 1: Use the solution of the Monge-Ampère equation to find a Hessian metric g with constant volume.

Step 2: Use Koszul and Calabi-Pogorelov (as above) to show that g is constant, and the connection on M is the Levi-Civita connection.

Step 3: Bieberbach's solution of Hilbert 18 implies that **all compact flat Riemannian manifolds are finite quotient of a torus.** ∎