Hölder continuity for potentials in hyperkähler dynamics

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Geometric structures on manifolds,
IMPA, 23.01.2025

Joint work in progress with Dimitri Korshunov

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M}\Big(|f| + |\nabla f| + ... + |\nabla^k f|\Big)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

REMARK: The pairing between forms and currents is denoted as $\alpha, \tau \mapsto \int_M \alpha\wedge\tau$. Using this notation, we interpret k forms on n-manifold as k-currents, that is, as functionals on $n-k$-forms.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p, q)-currents** are (p, q)-forms with coefficients in generalized functions **REMARK: In the literature, this is sometimes called $(n - p, n - q)$-currents**.

CLAIM: The Poincare and Poincare Dolbeault-Grothendieck lemma hold on (p, q)-currents, and **the d- and $\overline{\partial}$-cohomology are the same as for forms.**

Positive forms

DEFINITION: A positive (1,1)-form on a complex manifold is a form $\eta \in \Lambda^{1,1}_{\mathbb{R}}(M)$ which satisfies $\eta(x, Ix) \geqslant 0$ for any $x \in TM$.

REMARK: "French positivity". For French, "positive" is the same as "non-negative" for the rest of the world. We will call functions "non-negative" if they are $\geqslant 0$, but if these functions are considered as 0-forms, we have to say they are "positive". **Please don't be confused!**

CLAIM: Let α be a positive function, and u a (1,0)-form. Then $-\sqrt{-1}\,\alpha u \wedge \overline{u}$ is a positive (1,1)-form. Moreover, **any positive form is obtained as a linear combination of such (1,1)-forms.**

Proof: Using the normal form of a positive (1,1)-form on a complex vector space (sometimes known as "polar decomposition"), we find that any positive (1,1)-form on an almost complex manifold can be locally represented as $\sum_i -\sqrt{-1}\,\alpha_i u_i \wedge \overline{u}_i$, where $\alpha \geqslant 0$ are non-negative functions, and $u_i \in \Lambda^{1,0}(M)$ an orthonormal frame. ■

Positive currents

REMARK: Positive generalized functions are all C^0-continuous as functionals on $C^\infty M$. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

DEFINITION: The cone of positive $(1,1)$-currents is generated by $-\sqrt{-1}\alpha u \wedge \overline{u}$, where α is a positive generalized function (that is, a measure), and u a (1,0)-form.

REMARK: This is equivalent to the following definition (the equivalence is a foundational result of theory of currents, found in both textbooks of Demailly).

DEFINITION: A $(1,1)$-current α is called **positive** if $\int_M \alpha \wedge \tau \geqslant 0$ for any positive $(n-1, n-1)$-form τ with compact support.

Positive currents: compactness theorem

DEFINITION: A **mass** of a positive (1,1)-current η on a Hermitian n-manifold (M,ω) is a measure $\eta\wedge\omega^{n-1}$. **It is non-negative, and positive if $\eta\neq 0$.**

Theorem: The space of positive $(1,1)$-currents with bounded mass is (weakly) compact.

Proof: Follows from precompactness of the space of bounded measures in weak-*-topology. ■

Rigid currents

DEFINITION: A nef class is a limit of Kähler $(1,1)$-classes in $H^{1,1}(M)$.

DEFINITION: A nef current is current obtained as a limit of positive, closed $(1,1)$-forms.

REMARK: All nef classes can be represented by nef currents (by compactnes).

DEFINITION: A nef class is called **rigid** if it has a unique positive, closed representative in the space of currents.

THEOREM: (Sibony, Soldatenkov, V.)
Let η be a nef class on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$. Assume that $\int_M \eta^{2n} = 0$, η is not proportional to a rational class, and the Pocard rank of M is not maximal. **Then the nef current representing η is rigid.**

CONJECTURE: Such currents **have Hölder continuous potentials,** that is, locally obtained as $dd^c f$, where f is a Hölder continuous function.

Motivation

Why do we care?

1. Rigid currents are **unique representatives with very special properties.** Unique things are always interesting.

2. Let T be a hyperbolic automorphism of a K3, η_+ its rigid current, η_- the rigid current of T^{-1}. Then $\eta_+ \wedge \eta_-$ is well defined (because both currents are nef and have bounded local potentials) and gives a **"maximal entropy measure"** (**Sinai-Ruelle-Bowen measure**) which is very important in dynamics and algebraic geometry (cf. **Lyubich theorem**). This result partially generalizes to other hyperkähler manifolds.

3. Let $[\eta]$ be a rational class on a boundary of the Kähler cone, $q(\eta, \eta) = 0$. It is no longer rigid; however, if its "Lelong numbers" (numbers measuring how singular this current is) vanish, which brings us very close to the **"SYZ conjecture"**, which is one of the central conjectures of hyperkähler geometry. Currents with bounded potential are **"really, really, really non-singular"** and have vanishing Lelong numbers; **understanding the Lelong numbers of rigid currents bring us closer to SYZ.**

Hyperbolic automorphisms

DEFINITION: An automorphism of a Calabi-Yau manifold is called **hyperbolic** if it acts on $H^{1,1}(M)$ with the largest real eigenvalue $\alpha \in \mathbb{R}$, $\alpha > 1$, and the corresponding eigenspace is 1-dimensional.

REMARK: If M is hyperkähler, **the corresponding eigenspace is 1-dimensional.** Indeed, the BBF form has signature $(1, n)$ on $H^{1,1}(M)$, and an isometry preserving a form of signature $(1, n)$ has at most 2 eigenvalues α_1, α_2 which satisfy $|\alpha_i| \neq 0$.

THEOREM: (Amerik, V.) Every hyperkähler manifold with $b_2 > 4$ **has a deformation which admits a hyperbolic automorphism.**

DEFINITION: A class $v \in H^{1,1}(M)$ on a hyperkähler manifold is called **dynamic** if M admits a hyperbolic automorphism T, and $[v]$ is its eigenvector with eigenvalue $\alpha > 1$.

Rigidity of dynamic currents

THEOREM: (Cantat, Dinh-Sibony)
Let $[v] \in H^{1,1}(M)$ be a dynamic class on a Calabi-Yau manifold. **Then $[v]$ is nef and rigid.**

Proof. Step 1: Let $v := \lim \frac{T^n \omega}{\alpha^n}$ be a current representing v. **The current v is nef.** Indeed, $\lim \frac{T^n \omega}{\alpha^n} = v$ for any $\omega \notin V$, where $V \subset H^{1,1}(M, \mathbb{R})$ is a subspace of positive codimension. Taking ω Kähler (the Kähler cone is open, hence we can assume that $\omega \notin V$), we obtain that v is nef.

Step 2: It remains to prove **uniqueness of the positive closed representative of $[v]$.** For any two positive representatives η_1, η_2, one has $\eta_1 - \eta_2 = dd^c\psi$ by dd^c-lemma; the set K of such ψ is compact, because the set of representatives of $[v]$ is compact. Adjusting the constant, may also assume that $\int_M \psi \operatorname{Vol} = 0$ for all such representatives $\psi \in K$, where Vol is an automorphism invariant volume form on M, obtained by taking a section of K_M and multiplying with its complex conjugate. By construction, $T^*K = \alpha K$.

Step 3: Each $\psi \in K$ is locally integrable (being a difference of two plurisubharmonic functions which are integrable). Consider the number $\sup_{\psi_1, \psi_2 \in K} \int_M |\psi_1 - \psi_2|$. **It is finite, because K is compact.** Since $T^*K = \alpha K$, we obtain $\alpha^n \sup_{\psi_1, \psi_2 \in K} \int_M |\psi_1 - \psi_2| \operatorname{Vol} = \sup_{\psi_1, \psi_2 \in K} \int_M |T^n\psi_1 - T^n\psi_2| \operatorname{Vol}$. This is impossible, because $\int_M |T^n\psi_1 - T^n\psi_2| \operatorname{Vol} = \int_M |\psi_1 - \psi_2| \operatorname{Vol}$. ■

Hölder continuous functions

DEFINITION: Let f be a function on a Riemannian manifold M, and d the metric on M, and $\alpha \in]0,1]$ a real number We say that f is **Hölder continuous** or **Hölder A, α-continuous** if $\sup_{x,y\in M} \frac{|f(x)-f(y)|}{d(x,y)^\alpha} < A$ is finite.

REMARK: When $\alpha = 1$, Hölder condition is equivalent to Lipschitz. It is also one of the "uniform continuity" conditions, which ensures that **a family of bounded Holder continuous functions is precompact in uniform topology (Arzelá-Ascoli).**

REMARK: The Hölder condition interpolates between continuity (C^0) and smoothness (which is more or less the same as Lipschitz), denoted C^1. This is why **the space of Holder continuous functions with exponent α is denoted C^α.** We will use the notation $C^{\alpha,A}$, when we need to fix both constants.

REMARK: Let $p := \alpha^{-1}$. **Then the Hölder A, α-condition means that $\sup_{x,y\in M} \frac{|f(x)-f(y)|^p}{d(x,y)} < A^p$ is bounded.**

Hölder continuity and diffeomorphisms

We are interested in Hölder condition because of the following observation.
Claim 1: Let M be a Riemannian manifold, and $\Psi \in \operatorname{Diff}(M)$ a diffeomorphism such that the norm $\|D\Psi\|$ of its differential is bounded by B, and let $\lambda \in]0,1[$ be a real number. Consider the number $\mu := -\frac{\log\lambda}{\log B}$ **Then $\lambda\Psi(f) \in C^{\alpha,A}$ for any $f \in C^{\alpha,A}$ and any $\alpha < \mu$.**
Proof: Set $p = \alpha^{-1}$. We need to show that this quantity is bounded: $\sup_{x,y\in M} \frac{|f(\Psi^{-1}x)-f(\Psi^{-1}y)|^p}{d(x,y)} < A^p$. However,

$$\sup_{x,y\in M} \frac{|\lambda f(\Psi^{-1}x) - \lambda f(\Psi^{-1}y)|^p}{d(x,y)} = \sup_{x,y\in M} \frac{|\lambda f(x) - \lambda f(y)|^p}{d(\Psi x, \Psi y)} \leqslant \sup_{x,y\in M} B\lambda^p \frac{|f(x) - f(y)|^p}{d(x,y)}.$$

This implies that $\lambda\Psi(f) \in C^{\alpha,A}$ whenever $f \in C^{\alpha,A}$ and $B\lambda^p \leqslant 1$. The last is translated to $\log B + p\log\lambda < 0$, or, equivalently, $p \geqslant -\frac{\log B}{\log\lambda}$, which gives $\alpha \leqslant -\frac{\log\lambda}{\log B}$. ■
REMARK: In other words, the map $f \mapsto \lambda\Psi(f)$ preserves the Hölder continuity. For instance, if f is bounded, **the series $\sum_{i=0}^\infty a_i\lambda^i\Psi^i(f)$ would converge to a Hölder continuous function when the series $\sum a_i$ is absolutely convergent.** The pointwise convergence follows from the convergence of $\sum a_i$ and boundedness of f, and the Hölder condition for the limit follows from Arzelá-Ascoli.

Hölder continuity and hyperbolic automorphisms

THEOREM: (Cantat, Dinh-Sibony) Let M be a compact Calabi-Yau manifold, $\gamma \in \operatorname{Diff}(M)$ a hyperbolic automorphism, and η the rigid current associated with γ, defined by $\gamma^*\eta = \lambda\eta$, where $\lambda > 1$ is an eigenvalue of γ on $H^{1,1}(M)$. In this situation, η can be written locally as $\eta = dd^c\psi$, where **the local potential ψ is Hölder continuous.**

Proof. Step 1: Fix a Calabi-Yau metric ω on M, and let $P: \ H^{1,1}(M) \longrightarrow \Lambda^{1,1}(M)$ denote the harmonic representative. Then for each $x \in H^{1,1}(M)$, the classes $\gamma^* P(X)$ and $P(\gamma(x))$ are homologous. Then dd^c-lemma **gives a unique function $u(x) \in C^\infty M$, $\int_M f \operatorname{Vol} = 0$, such that $dd^c(u(x)) = \gamma^* P(x) - P(\gamma^* x)$.** For any bounded set $K \subset H^{1,1}(M)$, the supremum $\sup_{x\in K} \sup |u(x)|$ is finite.

Hölder continuity and hyperbolic automorphisms (2)

Proof. Step 1: Fix a Calabi-Yau metric ω on M, and let $P: \ H^{1,1}(M) \longrightarrow \Lambda^{1,1}(M)$ denote the harmonic representative. Then for each $x \in H^{1,1}(M)$, the classes $\gamma^* P(X)$ and $P(\gamma(x))$ are homologous. Then dd^c-lemma **gives a unique function $u(x) \in C^\infty M$, $\int_M f \operatorname{Vol} = 0$, such that $dd^c(u(x)) = \gamma^* P(x) - P(\gamma^* x)$.** For any bounded set $K \subset H^{1,1}(M)$, the supremum $\sup_{x\in K} \sup |u(x)|$ is finite.
Step 2: We use notation γ^n for $(\gamma^*)^n$. Clearly,

$$\begin{aligned}\lambda^{-n}\gamma^n\omega - P(\gamma^n\omega) &= \sum_{k=1}^{n-1} \lambda^{-k}\gamma^k \Big[\lambda^{-n+k}\gamma^* P(\gamma^{n-k-1}\omega) - \lambda^{-n+k} P(\gamma^{n-k}\omega)\Big] \\ &= \sum_{k=1}^{n-1} \lambda^{-k} dd^c \left(\frac{u(P(\gamma^{n-k-1}\omega))}{\lambda^{n-k}}\right).\end{aligned}$$

Step 3: Let now $f_k := u\left(P\left(\frac{\gamma^{n-k-1}\omega}{\lambda^{n-k}}\right)\right)$. The set of classes $\frac{\gamma^{n-k-1}\omega}{\lambda^{n-k}}$ belongs to a compact subset of $H^{1,1}(M)$, because the corresponding sequence converges. Therefore, the functions f_k belong to the same compact family, and are Hölder (A,α)-continuous with fixed constants A, α. Claim 1 implies that for appropriate α, the functions $\lambda^{-k}\gamma^k f_k$ are also Hölder (A,α)-continuous, which implies that $\lambda^{-n}\gamma^n\omega - P(\gamma^n\omega) = \sum dd^c(\lambda^{-k}\gamma^k f_k)$ has (A,α)-continuous potential. **Passing to a limit as $n \to \infty$, we obtain that $\eta - P([\eta])$ has a (A,α)-continuous potential;** here $\eta = \lim_n \lambda^{-n}\gamma^n\omega$, and $P([\eta]) = \lim_n P(\lambda^{-n}\gamma^n\omega)$.
■

MBM classes

DEFINITION: **Negative class** on a hyperkähler manifold is $\eta \in H_2(M,\mathbb{R}) = H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$. It is **effective** if it is represented by a curve.

THEOREM: Let $z \in H_2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that z is of type (1,1) with respect to I and I' and $\operatorname{Pic}(M) = \langle z \rangle$, where $\operatorname{Pic}(M) = H^{1,1}(M,\mathbb{Z}) = H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$. Then $\pm z$ **is effective in** (M,I) $\Leftrightarrow$ **iff it is effective in** (M,I').

REMARK: From now on, we identify $H^2(M)$ and $H_2(M)$ using the BBF form. Under this identification, **integer classes in** $H_2(M)$ **correspond to rational classes in** $H^2(M)$ (the form q is not unimodular).

DEFINITION: A negative class $z \in H^2(M,\mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\operatorname{Pic}(M) = \langle z \rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

MBM classes and the shape of the Kähler cone

THEOREM: Let (M, I) be a hyperkähler manifold, and $S \subset H_{1,1}(M, I)$ the set of all MBM classes in $H_{1,1}(M, I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ | \ z \in S\}$ in $H^{1,1}(M, I)$. **Then the Kähler cone of (M, I) is a connected component of $\operatorname{Pos}(M, I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M, I)$ is a positive cone of (M, I). Moreover, for any connected component K of $\operatorname{Pos}(M, I) \backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M, I') birationally equivalent to (M, I), such that $\gamma(K)$ is a Kähler cone of (M, I').

REMARK: This implies that **MBM classes correspond to the faces of the Kähler cone.**

COROLLARY: A Kähler cone of (M, I) **is round if and only if the set of MBM classes in $H^{1,1}(M, I)$ is empty.**

THEOREM: (Amerik-V.) For any hyperkähler manifold M, **there exist a deformation (M, I) with round Kähler cone.** Moreover, if $b_2(M) > 5$, the Picard lattice of (M, I) can have signature $(1, 2)$ **(I. Frolov)**.

MBM classes and automorphisms

THEOREM: Let (M, I) be a hyperkähler manifold, and $\operatorname{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems. **Then** $\operatorname{Mon}(M)$ **is a finite index subgroup in** $O(H^2(M, \mathbb{Z}), q)$**, where** q **is BBF form.**

THEOREM: Let (M, I) be a hyperkähler manifold, $\operatorname{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\operatorname{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\operatorname{Mon}(M)$ preserving the Hodge decomposition. Denote by $\operatorname{Aut}_h(M, I)$ the image of the automorphism group in $GL(H^2(M, \mathbb{R}))$. **Then** $\operatorname{Aut}_h(M, I)$ **is a subgroup of** $\operatorname{Mon}_I(M)$ **preserving the Kähler cone** $\operatorname{Kah}(M, I)$**.**

REMARK: The kernel of the natural map $\operatorname{Aut}(M) \longrightarrow GL(H^2(M, \mathbb{R}))$ **is a finite group which is independent from the choice of** M **in its deformation class.** It consists of **"absolutely trianalytic"** automorphisms of M: automorphisms which are hyperkähler in all hyperkähler structures.

Automorphisms and lattices

COROLLARY: Let (M, I) be a hyperkähler manifold with round Kähler cone. **Then** $\operatorname{Aut}(M)$ **surjects to** $\operatorname{Mon}_I(M)$ **with finite kernel.** Moreover, its image has finite index in the subgroup of all elements in $O(H^2(M,\mathbb{Z}),q)$ preserving the Hodge decomposition.

DEFINITION: The **Neron-Severi lattice** of a Kähler manifold M is $H^{1,1}(M,\mathbb{Z}) = H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$. **The ample cone** $\operatorname{Kah}_{\mathbb{Q}}$ of M is $H^{1,1}(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ intersected with its Kähler cone.

REMARK: Rational points are dense in the ample cone.

COROLLARY: Let (M, I) be a hyperkähler manifold with round Kähler cone, and $\mathbb{P}\operatorname{Kah}_{\mathbb{Q}}$ the projectivization of its ample cone, identified with the positive cone in $H^{1,1}(M,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\mathbb{P}\operatorname{Kah}$ is a hyperbolic space, and $\operatorname{Aut}(M)$ **acts on** $\mathbb{P}\operatorname{Kah}$ **as a lattice.** If, in addition, $H^{1,1}(M,\mathbb{Z})$ contains no point x which satisfy $q(x,x) = 0$, **the quotient** $\frac{\mathbb{P}\operatorname{Kah}_{\mathbb{Q}}}{\operatorname{Aut}(M)}$ **is a compact hyperbolic orbifold.**

Quasi-isometry

DEFINITION: A map $f: X \longrightarrow Y$ is called **bi-Lipschitz with constant C**, or just **bi-Lipschitz**, if it is bijective, and both f and f^{-1} are C-Lipschitz (that is, satisfy $d(f(x), f(y)) \leqslant Cd(x, y)$). Two spaces X, Y are **bi-Lipschitz equivalent** if there exists a bi-Lipschitz map $f: X \longrightarrow Y$.

DEFINITION: The spaces X and Y are **quasi-isometric**, if X and Y are equipped with a ε-networks $X_\varepsilon \subset X$, $Y_\varepsilon \subset Y$ which are bi-Lipschitz equivalent.

EXAMPLE: Let Γ be a group, S_1 and S_2 its finite sets of generators, and Γ_1, Γ_2 the corresponding Cayley graphs. **Then Γ_1 is quasi-isometric to Γ_2.**

REMARK: When we are interested in metric spaces up to quasi-isometry, **we can speak of a group as of a metric space.**

THEOREM: (Milnor-Schwarz)
Let M be a compact Riemannian manifold, and $\tilde M$ its universal cover. **Then $\tilde M$ is quasi-isometric to $\pi_1(M)$.**

Gromov hyperbolic spaces

DEFINITION: A Gromov hyperbolic space is a geodesic metric space such that there exists $\delta \leqslant 0$ such that for any geodesic triangle, any side belongs to a δ-neighbourhood of the union of the other two.

THEOREM: ("Morse lemma") Let M, M' be quasi-isometric geodesic metric spaces. **Then M is Gromov hyperbolic iff M' is.**

DEFINITION: A group is **Gromov hyperbolic** or **word hyperbolic** if its Cayley graph is Gromov hyperbolic.

PROPOSITION: Let M be a complete Riemannian manifold with section curvature bounded from above by $C < 0$. **Then M is Gromov hyperbolic.**

COROLLARY: A fundamental group of a compact Riemannian manifold with section curvature bounded from above by $C < 0$ **is Gromov hyperbolic**.

Gromov hyperbolic boundary

DEFINITION: Let M be a Gromov hyperbolic space, and $\gamma(t)$ a geodesic ray which goes to ∞. Two such rays $\gamma(t), \gamma'(t)$ are called **equivalent** if $d(\gamma(t), \gamma'(t)) < const$. **Gromov boundary** ∂M of M is the set of equivalence classes of geodesic rays.

There is no metric on ∂M, but ∂M is equipped with **a natural topology and a conformal metric structure,** that is, a math metric up to a conformal factor.

DEFINITION: Fix a point p in a metric space M. Define **the Gromov product** $(x,y)_p$ of x, y as $(x,y)_p := 1/2(d(x,p) + d(y,p) - d(x,y)$. If $\gamma(t), \gamma'(t)$ are geodesic rays, define **the overlap** $p(\gamma|\gamma') := \lim_{t\to\infty} (\gamma(t), \gamma'(t))_p$.

REMARK: When all points on a Gromov hyperbolic boundary ∂M are connected by infinite geodesics (say, we are on a universal cover of a compact geodesic metric space, such as a Cayley graph or a Riemannian manifold), **the overlap satisfies $|p(\gamma|\gamma') - d(p, \gamma_\infty)| < 4\delta$, where γ_∞ is an infinite geodesic which is asymptotic to γ as $t \longrightarrow \infty$ and to γ' as $t \longrightarrow -\infty$.**

Visual distance

DEFINITION: A visual distance on ∂M with respect to $p \in M$ is a metric d on ∂M such that there are constants $C, \alpha > 0$ such that for any pair $\gamma(t), \gamma'(t)$ of rays leaving p, we have $C^{-1}e^{-\alpha p(\gamma|\gamma')} \leqslant d(\gamma, \gamma') \leqslant Ce^{-\alpha p(\gamma|\gamma')}$.

REMARK: A visual distance always exists; it depends on the choice of p, but this dependence is (generally speaking) conformal. On the hyperbolic space $\mathbb{H}^n$, **the visual distance is the angle between geodesics γ, γ';** this implies that $\partial\mathbb{H}^n$ is a sphere with the round metric. Also, **∂M is conformal to $\partial M'$ when M, M' are quasi-isometric Gromov hyperbolic spaces.**

Hölder continuity and the geometric group theory

THEOREM: Let (M,ω) be a compact hyperkähler manifold with round Kähler cone, $[\eta] \in \partial\,\mathrm{Kah}_{\mathbb{Q}}$ a class on a boundary such that $q([\eta],[\eta]) = 0$, and η the corresponding rigid (1,1)-current. **Then η locally has a Hölder continuous potential**.

Proof. Step 1: Let H be a compact hyperbolic manifold, and $\mathbb{H}^n$ its universal cover. Let $\{s_1, ..., s_m\}$ be a set of generators in $\pi_1(H)$. The geodesics in the Cayley graph of $\pi_1(M)$ are encoded by a sequence of words $W_0, W_1, ..., W_n, ...$ such that $W_n = W_{n-1} s_{i_n}$.

Step 2: Clearly, the hyperbolic boundary of $\mathbb{H}^n$ is a sphere with the round metric; since $\pi_1(H)$ is quasi-isometric to H, the same is true for $\pi_1(M)$. Choosde a set of generators $\{s_1, ..., s_m\}$ for $\mathrm{Aut}(M)$. Applying Step 1, we obtain that **for any $x \in \partial\mathbb{P}\,\mathrm{Kah}\,M$ there exists a geodesic in $\mathrm{Aut}(M)$ encoded by a sequence $W_0, W_1, ..., W_n, ...$ such that $[\eta] = \lim_n \lambda_n W_n([\omega])$, where ω is a reference Kähler metric, and λ_n a sequence of real numbers**.

Hölder continuity and the geometric group theory (2)

Step 3: Applying this to $[\eta]$ and using rigidity of the current. we obtain that $\eta = \lim_n \lambda_n W_n(\omega)$, where the limit is taken in currents. Denote by $\lambda(W_n)$ the number $\lambda(W_n) := q(W_n\omega, \omega)^{-1}$. Then $\lambda(W_n)W_n(\omega)$ converges to to a rigid current proportional to η. We understand the geodesic $\{W_i\}$ as an infinite sequence (word) in the alphabeth $\{s_i\}$, and let $W_{k,n}$ be a subsequence which starts in k and ends in n. Define $\lambda(W_{k,n}) := \frac{\lambda(W_n)}{\lambda(W_k)}$. We have a formula similar to given above,

$$\lambda(W_n)(W_n\omega - P(W_n\omega) = \sum_{k=1}^{n-1} \lambda(W_k)W_k\lambda(W_{k,n}) \left[s_{k+1}P(W_{k+1,n}\omega) - P(W_{k,n}\omega)\right]$$

$$= \sum_{k=1}^{n-1} \lambda(W_k)W_k \left[dd^c\lambda(W_{k,n})u_{s_{k+1}}(W_{k+1,n}\omega)\right]$$

where $u_s(x)$ is a function with $\int_M u_s(x)\,\mathrm{Vol} = 0$ given by $dd^c u_s(x) = s(P(x)) - P(sx)$. This sequence converges to a Hölder continuous limit by the same argument as above (we need to use the exponential estimates $\lambda(W_i) \sim \lambda^i$, with some fixed constant λ depending on the geodesic $\{W_i\}$; these estimates follow from standard formulas in hyperbolic geometry). ■