Moser's lemma for C-symplectic structures

Misha Verbitsky

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Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ **is a vector space, and** Symp **an infinite-dimensional (Fréchet) manifold.**

DEFINITION: Let Diff_0 be the group of isotopies of M, that is, **the connected component of the diffeomorphism group.** **Teichmüller space of symplectic structures on M** is defined as the quotient space $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

REMARK: Let $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ be the mapping class group of M. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$ **is identified with the set of symplectic structures up to diffeomorphism.**

Moser's theorem

DEFINITION: Let M be compact. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965) The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Proof of Moser's theorem: The period map $P: U \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion of infinite-dimensional smooth manifolds. By Moser's lemma, the fibers of P are 0-dimensional. **Therefore, P is locally a diffeomorphism.** ■

The proof of Moser's lemma

Moser's lemma: Let ω_t, $t\in[0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t]\in H^2(M)$ is constant in t. **Then there exists a smooth family $\Psi_t\in \operatorname{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^*\omega_0=\omega_t$.**

Proof: We construct Ψ_t as a solution of the equation $\frac{d\Psi_t}{dt}=X_t$, where $X_t\in TM$ is a vector field depending on $t\in[0,1]$.

Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. This gives $\frac{d\omega_t}{dt}=d\eta_t$, where $\eta_t\in\Lambda^1(M)$ smoothly depends on $t\in[0,1]$. Let X_t be the vector field which satisfies $\omega_t \lrcorner X_t=\eta_t$. **Cartan's formula gives** $\operatorname{Lie}_{X_t}\omega_t = d(\omega_t\lrcorner X_t)=d\eta_t=\frac{d\omega_t}{dt}$.

Step 2: Let Ψ_t be the flow of diffeomorphisms obtained by integrating X_t. By construction, $\operatorname{Lie}_{X_t}\omega_t=\frac{d\omega_t}{dt}$. Integrating it in t, we obtain

$$\Psi_1^*\omega_0=\int_0^1\Psi_t\operatorname{Lie}_{X_t}\omega_t dt=\int_0^1\frac{d\omega_t}{dt}dt=\omega_1.$$

■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$. **The eigenvalues of this operator are** $\pm\sqrt{-1}$ **.** The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the standard one.

CLAIM: (the Hodge decomposition determines the complex structure)
Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles** $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ **satisfying** $\dim_{\mathbb{C}} T^{0,1}M = n$ **and** $T^{0,1}M \cap TM = 0$ (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\Big|_{T^{1,0}M} = \sqrt{-1}$ and $I\Big|_{T^{0,1}M} = -\sqrt{-1}$. ■

Holomorphically symplectic manifolds

DEFINITION: Let (M, I) be a complex manifold, and $\Omega \in \Lambda^2(M, \mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$.

REMARK: The equation $\Omega(IX, Y) = \sqrt{-1}\Omega(X, Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore, $\ker(\Omega) = T^{0,1}M$.**

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M, I). **Then I is determined by Ω uniquely.**

C-symplectic structures

DEFINITION: (Bogomolov, Deev, V.) Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where

$$\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \mid \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re} \Omega$ is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula

$$d\Omega(X_1, X_2, X_3) = \frac{1}{6} \sum_{\sigma \in \Sigma_3} (-1)^{\tilde{\sigma}} \operatorname{Lie}_{X_{\sigma_1}} \Omega(X_{\sigma_2}, X_{\sigma_3}) + (-1)^{\tilde{\sigma}} \Omega([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3})$$

which gives, for all $X, Y \in T^{0,1}M$, and any $Z \in TM$,

$$d\Omega(X, Y, Z) = \Omega([X, Y], Z),$$

implying that $[X, Y] \in T^{0,1}M$. ■

Period map for holomorphically symplectic manifolds

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\operatorname{CTeich} := \frac{\operatorname{CSymp}}{\operatorname{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\operatorname{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0, 1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Holomorphically symplectic Moser's lemma

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M,\mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^*\Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$, we have (like in the proof of Moser's lemma)

$$V_{t_1}^*\Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t}\Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow such that $\frac{dV_t}{dt} = X_t$. **It remains to find the family $X_t \in T_{\mathbb{R}}M$.**

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb{R}} T_{\mathbb{R}}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective** (an exercise).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Holomorphically symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R}M$ such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$.

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb R} T_{\mathbb R}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \xrightarrow{\wedge\Phi} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ implies that $\Omega_t' \in \Lambda^{1,1}(M, I_{\Omega_t}) + \Lambda^{2,0}(M, I_{\Omega_t})$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Holomorphically symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

COROLLARY: Let CSymp be the space of all C-symplectic structures with C^∞-topology. Denote by Teich_C the corresponding Teichmüller space, $\operatorname{Teich}_C := \frac{\text{CSymp}}{\operatorname{Diff}_0(M)}$. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_C \longrightarrow H^2(M, \mathbb{C})$ mapping Ω to its cohomology class. **Then Per is locally a homeomorphism to its image.**

Proof: All fibers of Per are 0-dimensional. ■

Local Torelli theorem for a K3 surface

REMARK: In real dimension 4, C-symplectic form **is a pair ω_1, ω_2 of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof: Next slide

REMARK: This theorem locally identifies $H^{1,1}(M)$ with the neighbourhood Ω in the C-symplectic Teichmüller space, proving that it is smooth and $b_2 - 2$-dimensional. **This proves the local Torelli theorem for K3.**

REMARK: The proof of this theorem is done using the same argument as used to prove the Maurer-Cartan equation, central to Kuranishi theory. Indeed, **the equation (*) we are going to solve below is a version of Maurer-Cartan, adopted and simplified for the C-symplectic structures.**

Local Torelli theorem for K3 (2)

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof. Step 1: Since $(\Omega + \rho)^2 = \rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, this form is (almost) C-symplectic. **To prove that it is C-symplectic, we need to find ρ such that that $d\rho = 0$.**

Step 2: From Hodge to de Rham isomorphism, we obtain that the cohomology class $[u]$ of $\Omega + \rho$ is equal to $[\Omega + \eta + u^{0,2}]$. Since M is K3, we have $H^{0,2}(M) = \mathbb{C}[\overline{\Omega}]$, which gives $[u^{0,2}] = \lambda[\overline{\Omega}]$, for some $\lambda \in \mathbb{C}$. since $(\Omega + \rho)^2 = 0$, this gives $[\Omega \wedge u^{0,2}] = [\eta]$, Then $\lambda = -\frac{[\eta^2]}{[\Omega \wedge \overline{\Omega}]}$. **We proved that the cohomology class of $\Omega + \rho$ is uniquely determined by $[\eta^2]$.**

Local Torelli theorem for K3 (3)

Below, we need the following version of $\partial\overline{\partial}$**-lemma**: **for any** $(1,2)$**-form** α**, which is** ∂**-exact and** $\overline{\partial}$**-closed,** $\alpha = \overline{\partial}\beta$**, where** β **is** ∂**-exact.**

Step 3: Let Λ_Ω be contraction with the (2,0)-bivector associated with Ω. This operation clearly commutes with $\overline{\partial}$. Then $\rho^{1,1}\wedge\rho^{1,1} = -\Omega\wedge\rho^{0,2}$ is equivalent to $\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\rho^{0,2}$. **To solve the equation** $d\rho = 0$**, we solve the equivalent equation, which is a version of Maurer-Cartan**

$$\partial\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\overline{\partial}\rho^{1,1}, \qquad \partial\rho^{1,1} = 0. \qquad (*)$$

Let γ_0 be the harmonic (1,1)-form representing $[\eta]$. We solve the equation (*) inductively by taking

$$\overline{\partial}\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right). \quad (**)$$

Such γ_n is found using $\partial\overline{\partial}$-lemma, because the RHS of (**) is ∂-exact and $\overline{\partial}$-closed, which is clear because $\overline{\partial}$ commutes with Λ_Ω. Since $\overline{\partial}\sum_i\gamma_i = \partial\Lambda_\Omega\left(\sum_{i,j}\gamma_i\wedge\gamma_j\right)$, the sum $\rho^{1,1} := \sum\gamma_i$ is a solution of (*).

Step 4: Since γ_i, $i>0$ are ∂-exact, the ∂-cohomology class of γ is $[\gamma_0] = [\eta]$.

■