Moser's lemma for holomorphically symplectic structures

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Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a vector space, and Symp an infinite-dimensional (Fréchet) manifold.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

REMARK: Let $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ be the mapping class group of M. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, **is identified with the set of symplectic structures up to diffeomorphism.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Let M be compact. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Proof of Moser's theorem: The period map $P: U \longrightarrow H^2(M,\mathbb{R})$ is a smooth submersion of infinite-dimensional smooth manifolds. By Moser's lemma, the fibers of P are 0-dimensional. **Therefore, P is locally a diffeomorphism.** ∎

The proof of Moser's lemma

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a smooth family $\Psi_t \in \operatorname{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.**

Proof: We construct Ψ_t as a solution of the equation $\frac{d\Psi_t}{dt} = X_t$, where $X_t \in TM$ is a vector field depending on $t \in [0,1]$.

Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. This gives $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$ smoothly depends on $t \in [0,1]$. Let X_t be the vector field which satisfies $\omega_t \lrcorner X_t = \eta_t$. **Cartan's formula gives $\operatorname{Lie}_{X_t} \omega_t = d(\omega_t \lrcorner X_t) = d\eta_t = \frac{d\omega_t}{dt}$.**

Step 2: Define Ψ_t using $\frac{d\Psi_t}{dt} = X_t$. Integrating in t the equation $\operatorname{Lie}_{X_t} \omega_t = \frac{d\omega_t}{dt}$, we obtain

$$\Psi_1^* \omega_0 = \int_0^1 \operatorname{Lie}_{X_t} \omega_t dt = \int_0^1 \frac{d\omega_t}{dt} dt = \omega_1.$$

■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

REMARK: The "usual definition": complex structure is an atlas on a manifold with differentials of all transition functions in $GL(n,\mathbb{C})$.

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

REMARK: An almost complex structure I **is uniquely determined by a subbundle $B \subset TM \otimes_{\mathbb{R}} \mathbb{C}$** such that $TM \otimes_{\mathbb{R}} \mathbb{C} = B \oplus \overline{B}$. Then we write $I = \sqrt{-1}$ on B and $I = -\sqrt{-1}$ on $\overline{B}$.

Holomorphically symplectic manifolds

DEFINITION: Let (M, I) be a complex manifold, and $\Omega \in \Lambda^2(M, \mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$.

REMARK: The equation $\Omega(IX, Y) = \sqrt{-1}\Omega(X, Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore,** $\ker(\Omega) = T^{0,1}M$**.**

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M, I). **Then I is determined by Ω uniquely.**

Holomorphically symplectic forms and complex structures

CLAIM: Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying** $\dim_{\mathbb{C}} T^{0,1}M = n$ **and** $T^{0,1}M \cap TM = 0$ (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\Big|_{T^{1,0}M} = \sqrt{-1}$ and $I\Big|_{T^{0,1}M} = -\sqrt{-1}$. ■

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a smooth, complex-valued, non-degenerate 2-form on a $4n$-dimensional real manifold. Assume that $\Omega^{n+1} = 0$. Consider the bundle

$$T^{0,1}_\Omega(M) := \{v \in TM \otimes \mathbb{C} \ \ | \ \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M)$ satisfies assumptions of the claim above, hence **defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable.**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and it is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula. ■

Period map for holomorphically symplectic manifolds

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and Symp_S the space of all holomorphically symplectic forms. The quotient $\mathsf{Teich}_S := \frac{\mathsf{Symp}_S}{\mathsf{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\mathsf{Teich}_S \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

THEOREM: Let (M, I_t, Ω_t), $t \in [0, 1]$ be a family of holomorphic symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M, I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \mathsf{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Holomorphically symplectic Moser's lemma

THEOREM: Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of holomorphic symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M,\mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^*\Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$, we have

$$V_{t_1}^*\Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t}\Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow such that $\frac{dV_t}{dt} = X_t$. **It remains to find $X_t \in T_{\mathbb{R}}M$.**

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb{R}} T_{\mathbb{R}}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective** (an exercise).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Holomorphically symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R} M$ such that $\operatorname{Lie}_{X_t} \Omega_t = \frac{d}{dt}\Omega_t$**.**

Step 2: The contraction map $\Lambda^{2,0} M \otimes_{\mathbb R} T_{\mathbb R} M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \xrightarrow{\wedge\Phi} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ implies that $\Omega_t' \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Holomorphically symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

Family of Lagrangian subvarieties

Lemma 1: Let (M, I_t, Ω_t), $t \in [0,1]$ be a smooth family of holomorphically symplectic manifolds (not necessarily compact), with all Ω_t exact, and $C \subset (M, I_t)$ holomorphic Lagrangian subvarieties. Assume that $H^{0,1}(M, I_t) = 0$. **Then C has a family U_t of open neighbourhoods in M such that (U_t, I_t, Ω_t, C) is trivialized by a flow of diffeomorphisms.**

Proof. Step 1: Find the vector field X_t as in the proof of Moser's lemma, in such a way that $d(\Omega_t \lrcorner X_t) = \frac{d}{dt}\Omega_t$. This is possible to do because $H^{0,1}(M, I_t) = 0$. **We want to modify X_t in such a way that it is tangent to C.** Let $\alpha_t = \Omega_t \lrcorner X_t$; this form satisfies $d\alpha_t = \frac{d}{dt}\Omega_t$. Since C is Lagrangian, X_t is tangent to C if and only if $\alpha_t|_C = 0$. However, $\frac{d}{dt}\Omega_t|_C = 0$, hence $\alpha_t|_C$ is closed. Shrinking M if necessary, we can assume that the restriction $H^1(M) \longrightarrow H^1(C)$ is surjective. Then we replace α_t by $\alpha_t - \gamma_t$, where γ_t is closed on M and satisfies $(\alpha_t - \gamma_t)|_C = 0$. Now we replace X_t by Y_t such that $\Omega_t \lrcorner Y_t = \alpha_t - \gamma_t$. This is another solution of Moser's equation $d(\Omega_t \lrcorner Y_t) = \frac{d}{dt}\Omega_t$, but now Y_t is tangent to C.

Step 2: Since C is compact, **Y_t can be integrated to a flow of diffeomorphisms in a neighbourhood of C** mapping (I_0, Ω_0) to (I_t, Ω_t), $t \in [0,1]$.

■

Weinstein tubular neighbourhood theorem for holomorphically symplectic manifolds

COROLLARY: Let (M, I, Ω) be a holomorphically symplectic manifold (not necessarily compact) with Ω exact, and $C \subset (M, I)$ a compact holomorphic Lagrangian subvariety. Assume that $H^{0,1}(M, I_t) = 0$ and the restriction map $H^1(M) \longrightarrow H^1(C)$ is surjective. **Then C has a neighbourhood which is isomorphic to a neighbourhood of C in T^*C as a holomorphically symplectic manifold.**

Proof: Choose a tubular neighbourhood U of C (in smooth category), identifying U and a small neighbourhood of C in T^*C. Let H_t be a homothety of T^*C mapping $v \in T^*C$ to tv. We may assume that $H_t(U) \subset U$. **The family of holomorphic symplectic forms $t^{-1}H_t^*\Omega$ converges to the standard holomorphic symplectic form on T^*C.** Now, Lemma 1 is used to trivialise this family in a neighbourhood of C. ■

REMARK: **Weinstein tubular neighbourhood theorem fails** when C is a fiber of a holomorphic Lagrangian fibration on a hyperkähler manifold (say, on an elliptic K3 surface).