Weinstein normal form for holomorphically Lagrangian submanifolds

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Weinstein normal form for holomorphic Lagrangian submanifolds

DEFINITION: Let $X \subset Y$ be a complex subvariety in a complex manifold Y. We say that X **can be bimeromorphically contracted** if there exists a morphism of complex varieties $Y \longrightarrow Y_1$ mapping X to a point and bijective on $Y\backslash X$.

The main result today.

THEOREM: (Amerik, V.) Let (M, I, Ω) be a holomorphically symplectic manifold (not necessarily compact) with Ω exact, and $E \subset (M, I)$ a compact holomorphic Lagrangian submanifold. Assume that E can be bimeromorphically contracted. **Then E is isomorphic to $\mathbb{C}P^n$.** Moreover, **E has a neighbourhood which is biholomorphically symplectomorphic to a neighbourhood of $\mathbb{C}P^n$ in $T^*\mathbb{C}P^n$.**

REMARK: Weinstein tubular neighbourhood theorem fails when E is a fiber of a holomorphic Lagrangian fibration on a hyperkähler manifold (say, on an elliptic K3 surface). Indeed, the normal bundle NE is trivial, but the elliptic curve in the elliptic family varies, hence **its neighbourhood cannot be isomorphic to $T^*E = E \times \mathbb{C}$.**

Moser's lemma

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a smooth family $\Psi_t \in \operatorname{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.**

Proof: We construct Ψ_t as a solution of the equation $\frac{d\Psi_t}{dt} = X_t$, where $X_t \in TM$ is a vector field depending on $t \in [0,1]$.

Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. This gives $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$ smoothly depends on $t \in [0,1]$. Let X_t be the vector field which satisfies $\omega_t \lrcorner X_t = \eta_t$. **Cartan's formula gives** $\operatorname{Lie}_{X_t} \omega_t = d(\omega_t \lrcorner X_t) = d\eta_t = \frac{d\omega_t}{dt}$.

Step 2: Define Ψ_t using $\frac{d\Psi_t}{dt} = X_t$. Integrating in t the equation $\operatorname{Lie}_{X_t} \omega_t = \frac{d\omega_t}{dt}$, we obtain

$$\Psi_1^* \omega_0 = \int_0^1 \operatorname{Lie}_{X_t} \omega_t dt = \int_0^1 \frac{d\omega_t}{dt} dt = \omega_1.$$

■

Holomorphically symplectic manifolds

DEFINITION: Let (M,I) be a complex manifold, and $\Omega \in \Lambda^2(M,\mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$.

REMARK: The equation $\Omega(IX,Y) = \sqrt{-1}\Omega(X,Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX,Y) = \sqrt{-1}\,\Omega(X,Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore, $\ker(\Omega) = T^{0,1}M$.**

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M,I). **Then I is determined by Ω uniquely.**

C-symplectic structures

DEFINITION: (Bogomolov, Deev, V.) Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where

$$\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \ | \ \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re} \Omega$ is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}} M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula

$$d\Omega(X_1, X_2, X_3) = \frac{1}{6} \sum_{\sigma \in \Sigma_3} (-1)^{\tilde{\sigma}} \operatorname{Lie}_{X_{\sigma_1}} \Omega(X_{\sigma_2}, X_{\sigma_3}) + (-1)^{\tilde{\sigma}} \Omega([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3})$$

which gives, for all $X, Y \in T^{0,1}M$, and any $Z \in TM$,

$$d\Omega(X, Y, Z) = \Omega([X, Y], Z),$$

implying that $[X, Y] \in T^{0,1}M$. ■

C-symplectic Moser's lemma

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t} \Omega_t = \frac{d}{dt}\Omega_t$, we have (like in the proof of Moser's lemma)

$$V_{t_1}^* \Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t} \Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow such that $\frac{dV_t}{dt} = X_t$. **It remains to find the family $X_t \in T_{\mathbb{R}} M$.**

Step 2: The contraction map $\Lambda^{2,0} M \otimes_{\mathbb{R}} T_{\mathbb{R}} M \longrightarrow \Lambda^{1,0}(M)$ **is surjective** (an exercise).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t} \Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Holomorphically symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R} M$ such that $\operatorname{Lie}_{X_t} \Omega_t = \frac{d}{dt}\Omega_t$**.**

Step 2: The contraction map $\Lambda^{2,0} M \otimes_{\mathbb R} T_{\mathbb R} M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t} \Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \stackrel{\wedge\Phi}{\longrightarrow} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ implies that $\Omega_t' \in \Lambda^{1,1}(M, I_{\Omega_t}) + \Lambda^{2,0}(M, I_{\Omega_t})$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Holomorphically symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

C-symplectic Moser lemma for non-compact manifolds

As in the usual symplectic situation, the Moser argument can be also applied to non-compact manifold.

THEOREM: (Soldatenkov, V.) Let $\pi\colon \mathcal{X} \to \Delta$ be a smooth family of holomorphic symplectic manifolds (not necessarily compact) over the unit disc, trivial as a family of C^∞ manifolds. Denote by $\mathcal{X}_t = \pi^{-1}(t)$ its fiber, and let $\Omega_t \in H^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ be its holomorphic symplectic form, smoothly depending on t. Using the C^∞ trivialization to identify cohomology groups of the fibres, assume that the cohomology class of Ω_t does not depend on $t \in \Delta$, and $H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$. Let $K \subset \mathcal{X}_{t_0}$ be a compact subset. Then there exists an open neighbourhood $U \subset \Delta$ of $t_0 \in \Delta$, and an open subset $\tilde{U} \subset \pi^{-1}(U)$, with $K \subset \tilde{U}$, with the following property. The set $\tilde{U}$ is locally trivially fibred over U, **with all fibres $\tilde{U} \cap \pi^{-1}(t)$, $t \in U$ isomorphic as holomorphic symplectic manifolds.**

We will apply this result when K is a bimeromorphically contractible Lagrangian submanifold.

Family of Lagrangian subvarieties

Lemma 1: Let (M, I_t, Ω_t), $t \in [0,1]$ be a smooth family of C-symplectic manifolds (not necessarily compact), with all Ω_t exact, and $E_t \subset (M, I_t)$ holomorphic Lagrangian subvarieties. Assume that $H^{0,1}(M, I_t) = 0$. **Then E has a family U_t of open neighbourhoods in M such that (U_t, I_t, Ω_t, E) is trivialized by a flow of diffeomorphisms.**

Proof. Step 1: Find the vector field X_t as in the proof of Moser's lemma, in such a way that $d(\Omega_t \lrcorner X_t) = \frac{d}{dt}\Omega_t$. This is possible to do because $H^{0,1}(M, I_t) = 0$. **We want to modify X_t in such a way that it is tangent to E.** Let $\alpha_t = \Omega_t \lrcorner X_t$; this form satisfies $d\alpha_t = \frac{d}{dt}\Omega_t$. Since E is Lagrangian, X_t is tangent to E if and only if $\alpha_t|_E = 0$. However, $\frac{d}{dt}\Omega_t|_E = 0$, hence $\alpha_t|_E$ is closed. Shrinking M if necessary, we can assume that the restriction $H^1(M) \longrightarrow H^1(E)$ is surjective. Then we replace α_t by $\alpha_t - \gamma_t$, where γ_t is closed on M and satisfies $(\alpha_t - \gamma_t)|_E = 0$. Now we replace X_t by Y_t such that $\Omega_t \lrcorner Y_t = \alpha_t - \gamma_t$. This is another solution of Moser's equation $d(\Omega_t \lrcorner Y_t) = \frac{d}{dt}\Omega_t$, but now Y_t is tangent to E.

Step 2: Since E is compact, **Y_t can be integrated to a flow of diffeomorphisms in a neighbourhood of E** mapping (I_0, Ω_0) to (I_t, Ω_t), $t \in [0,1]$.

■

Weinstein normal form for holomorphic Lagrangian submanifolds

COROLLARY: Let (M,I,Ω) be a holomorphically symplectic manifold (not necessarily compact) with Ω exact, and $E\subset (M,I)$ a compact holomorphic Lagrangian subvariety. Assume that $H^{0,1}(M,I_t)=0$ and the restriction map $H^1(M)\longrightarrow H^1(E)$ is surjective. Assume, finally, that a neighbourhood of E can be smoothly deformed to a neighbourhood of the zero section in T^*E as a C-symplectic manifold with exact holomorphic symplectic form. **Then E has a neighbourhood which is isomorphic to a neighbourhood of E in T^*E as a holomorphically symplectic manifold.**

Proof: Now, Lemma 1 is used to trivialise this family in a neighbourhood of E. ■

Grauert-Riemenschneider theorem

The Grauert-Riemenschneider theorem takes care of the vanishing of $H^1(U, \mathcal{O}_U)$ in an appropriate neighbourhood U of a bimeromorphically contractible subvariety.

THEOREM: (Grauert-Riemenschneider)
Let $f: X \longrightarrow Y$ be a generically finite and surjective morphism of complex varieties. **Then $R^i f_*(K_X) = 0$,** where R^i is the derived direct image and K_X the canonical bundle.

Proof: *R. Lazarsfeld, Positivity in Algebraic Geometry. (Vol. I, page 257, Theorem 4.3.9.)* ■

COROLLARY: Let $E \subset M$ be a contractible Lagrangian subvariety of a holomorphic symplectic manifold. **Then any open neighbourhood of E in M contains a tubular neighbourhood $U \supset E$ such that $H^i(\mathcal{O}_E) = 0$** for any $i > 0$.

Proof: Let $f: M \longrightarrow M_1$ be the bimeromorphic contraction, mapping E to a point $x \in M_1$. Consider a Stein neighbourhood $V \ni x$, and let $U := f^{-1}(V)$. Since M is holomorphically symplectic, its canonical bundle is trivial, giving $K_M = \mathcal{O}_M$. This implies that $R^i f_*(\mathcal{O}_U) = 0$. The Grothendieck spectral sequence with E_2-table $H^j(R^i f_*(\mathcal{O}_U))$ converges to $H^{i+j}(\mathcal{O}_U)$, giving $H^k(\mathcal{O}_U) = H^k(f_* \mathcal{O}_U) = H^k(\mathcal{O}_V) = 0$ because V is Stein. ■

Contractible holomorphic Lagrangian submanifolds

THEOREM: Let E be a smooth subvariety of a projective symplectic variety X of dimension $2n$. Assume that E can be contracted to a point. **Then E is isomorphic to $\mathbb{C}P^n$.**

Proof: *Y. Hu and S.-T. Yau, HyperKahler Manifolds and Birational Transformations, Adv. Theor. Math. Phys. 6 (2002) 557-574*, Theorem C.

There are two problems with the proof. First, it is based on a (very important, interesting and insightful) paper *K. Cho, Y. Miyaoka, and N.I. Shepherd-Barron, Characterizations of projective spaces and applications and Applications to Complex Symplectic Manifolds. Adv. Stud. Pure Math., 2002: 1-88 (2002).* which is famous for being mostly wrong. Second, it assumes that X is projective, and we want a local result.

The part of Cho-Miyaoka-Shepherd-Barron which we need was fixed by S. Kebekus in *Kebekus, S. Characterizing the projective space after Cho, Miyaoka and Shepherd-Barron. In Complex geometry (Göttingen, 2000), pages 147-155.* Kebekus also assumes that X is projective, but his argument can be improved. In the end, we obtain

THEOREM: (Amerik, V.)
Let E be a compact complex submanifold of a holomorphic symplectic variety X. Assume that E can be contracted to a point. **Then E is isomorphic to $\mathbb{C}P^n$.**

Deformation to the normal cone

Let $X\subset M$ be a comples submanifold in a complex manifold. **Deformation to the normal cone** is a holomorphic deformation of a neighbourhood of $X\subset M$ over the disk such that its central fiber is the total space of the normal bundle NX, and the rest of the fibers are M. **It is obtained as follows.**

Let $X\subset M$ be a complex subvariety. Consider a product $M_1:=M\times\Delta$ of M with the disk Δ, and let $\tilde M_1$ be the blow-up of M_1 in $X\times\{0\}$. Denote by $\tilde\pi_1:\ \tilde{\mathcal X}\longrightarrow\Delta$ the blow-down composed with the projection. The preimage $\tilde\pi_1^{-1}(0)$ is a union of two irreducible components, the proper preimage of $M\times\{0\}$, denoted D_1, and the blow-up divisor, denoted D_2.

DEFINITION: The **deformation to the normal cone** is the complement $\tilde M:=\tilde M_1\backslash D_1$.

Clearly, the central fiber of the natural projection $\tilde M\longrightarrow\Delta$ is $D_2\backslash(D_1\cap D_2)$.

Deformation to the normal cone

CLAIM: When X is smooth, and M is a tubular neighbourhood of X in M, the complement $D_2\backslash(D_1 \cap D_2)$ is naturally isomorphic to NX, and **the "deformation to the normal cone" family $\tilde\pi : \tilde M \longrightarrow \Delta$ is locally trivial in smooth category.**

Proof. Step 1: The blow-up divisor $E = \mathbb{P}N_{M_1}X = \mathbb{P}(NX \oplus \mathcal{O}_X)$, and its intersection with D_1 is the set of all $l \in \mathbb{P}(NX \oplus \mathcal{O}_X)$ tangent to $M \times \{0\}$. We identify this intersection with $\mathbb{P}(NX)$. **This gives $D_2\backslash(D_1 \cap D_2) = \mathbb{P}(NX \oplus \mathcal{O}_X)\backslash\mathbb{P}(NX) = \operatorname{Tot}(NX)$.**

Step 2: Now, the tubular neighbourhood of $X \subset M$ is diffeomorphic to $\operatorname{Tot}(NX)$, **hence all fibers of $\tilde\pi : \tilde M \longrightarrow \Delta$ are diffeomorphic;** later today we will construct explicitly a vector field transversal the fibers of π and trivializing this family. ■

Deformation to the normal cone in holomorphic symplectic category

THEOREM 1: (Amerik, V.)
Let (M,Ω) be a holomorphically symplectic manifold, and $X\subset M$ a compact holomorphic Lagrangian submanifold, isomorphic to $\mathbb{C}P^n$ as shown above. Assume that M admits a proper, birational map which contracts X. Then there exists a smooth, holomorphic deformation of a neighbourhood of X in M over the disk Δ, such that its central fiber is biholomorphic to a neighbourhood of X in T^*X, the rest of the fibers are biholomorphic to a neighbourhood of X in M, **and the holomorphic symplectic form on T^*X can be smoothly extended to the holomorphic symplectic form on the rest of the fibers.**

REMARK: Note that this deformation in complex analytic category is already constructed: it is the "deformation to the normal cone" family. However, **to apply the C-symplectic Moser lemma, we need to have a smooth family of holomorphically symplectic forms on its fibers.**

Deducing Weinstein theorem from the deformation to normal cone

COROLLARY: Let (M, I, Ω) be a holomorphically symplectic manifold (not necessarily compact) with Ω exact, and $E \subset (M, I)$ a compact holomorphic Lagrangian submanifold. Assume that E can be bimeromorphically contracted. **Then E is isomorphic to $\mathbb{C}P^n$.** Moreover, **E has a neighbourhood which is biholomorphically symplectomorphic to a neighbourhood of $\mathbb{C}P^n$ in $T^*\mathbb{C}P^n$.**

Proof: Let $\tilde\pi: \tilde M \longrightarrow \Delta$ be the holomorphically symplectic deformation to the normal cone, constructed above. The fibers of $\tilde\pi$ are holomorphically symplectic and admit bimeromorphic contraction to a Stein manifold. By Grauert-Riemenschneider, the fibers $M_t := \tilde\pi^{-1}(t)$ satisfy $H^1(\mathcal{O}_{M_t}) = 0$, hence the assumptions of the non-compact version C-symplectic Moser lemma are satisfied, and **the family $\tilde\pi: \tilde M \longrightarrow \Delta$ is trivial in certain neighbourhoof of $X \times \Delta$.** ■

Deformation to the normal cone: preliminaries

We deduce the proof of Theorem 1 from the following propositions, which will be proven later.

Proposition 1: Let (M,Ω) be a holomorphically symplectic manifold, and $X\subset M$ a compact holomorphic Lagrangian submanifold. Assume that M admits a proper, birational map to a Stein variety which contracts X. **Then the natural map $H^0(\mathcal{O}_M)\longrightarrow H^0(\mathcal{O}_M/J_X^i)$ is surjective, for any $i\geqslant 0$**

Proposition 2: Let (M,Ω) be a holomorphically symplectic manifold, and $X\subset M$ a compact holomorphic Lagrangian submanifold. Assume that M admits a proper, birational map which contracts X. **Then for any section η_0 of $\Omega^1 M|_X$ there exists a closed 1-form η of $\Omega^1 M$ defined in a neighbourhood of X such that $\eta|_X=\eta_0$.**

Deformation to the normal cone: the proof

Proof of Theorem 1. Step 1: Let $\tilde\pi: \tilde M \longrightarrow \Delta$ be the deformation to the normal cone. Consider a fiberwise holomorphic symplectic form $\tilde\Omega := t^{-1}\Omega$ on $\tilde\pi^{-1}(\Delta\backslash 0)$, where t is the parameter in Δ. To prove Theorem 1 **it suffices to extend this form smoothly to a holomorphically symplectic form on the central fiber.**

Step 2: Replacing M by a smaller neighbourhood of X, **we will find a holomorphic 1-form $\theta \in \Omega^1 M$ such that $d\theta = \Omega$ and $\theta|_X = 0$.** This is done as follows. Since $\Omega|_X$ is exact, and a manifold is homotopy equivalent to its sufficiently small neighbourhood, it would suffice to prove Theorem 1 when Ω is exact, $\Omega = d\eta$. Shrinking a neighbourhood of X if necessarily and using Grauert-Riemenschneider again, we may also assume that $H^1(\mathcal{O}_M) = 0$. Since $\overline\partial\eta^{0,1} = 0$, this form is $\overline\partial$-exact: there exists $f \in C^\infty M$ such that $\overline\partial = \eta^{0,1}$. Replacing η by $\eta - df$, we obtain a $(1,0)$-form η such that $d\eta = \Omega$. This form is clearly holomorphic. Replacing η by $\theta := \eta - \eta_0$, where η_0 is constructed as in Proposition 2, we obtain that $d\theta = \Omega$ and $\theta|_X = 0$.

Deformation to the normal cone: the proof (2)

Step 3: Let $\tilde M \stackrel{\tilde\pi}{\longrightarrow} \Delta$ be the deformation to the normal cone family, and t the coordinate on Δ. **In Step 3, we prove that the fiberwise form $t^{-1}\Omega$ can be smoothly extended to the central fiber.**

Locally in X we can write X by a system of holomorphic equations $q_1 = q_2 = ... = q_n = 0$, and the holomorphically symplectic form as $\Omega = \sum_{i=1}^n dp_i \wedge dq_i$. The coordinates on the central fiber of the deformation to the normal cone family $\tilde M \stackrel{\tilde\pi}{\longrightarrow} \Delta$ are given by $p_1, ..., p_n, \tilde q_1, ..., \tilde q_n$. Trivializing the neighbourhood of $x \times \Delta \in X \times \Delta$ along Δ in the usual way, we write $\tilde q_i = t^{-1} q_i$: this is the standard way to write coordinates on the blow-up. Since $q_i = t\tilde q_i$, and q_i generate the ideal of X, a function on $\tilde M$ which vanishes on X is divisible by t.

Writing θ in these coordinates, and using $\theta|_X = 0$, we obtain

$$\theta = \sum_{i=1}^n u_i dq_i + v_i dp_i = \sum_{i=1}^n u_i t d\tilde q_i + u_i \tilde q_i dt + v_i dp_i$$

where $v_i|_X = 0$, hence divisible by t. The form $\theta_1 := \sum_{i=1}^n u_i t d\tilde q_i + v_i dp_i$ is divisible by t and its fiberwise differential is equal to $d\theta$, hence $d(t^{-1}\theta_1)$ is a smooth form which is equal to $t^{-1}\Omega$ on the general fibers of $\tilde M \stackrel{\tilde\pi}{\longrightarrow} \Delta$.

Deformation to the normal cone: the proof (3)

Step 4: It remains to show that $d(t^{-1}\theta_1)$ is non-degenerate in a neighbourhood of X in the central fiber of $\tilde{M} \xrightarrow{\tilde{\pi}} \Delta$. Writing $\Omega = \sum_{i=1}^n dp_i \wedge dq_i$ as above and passing to the coordinates $q_i = t\tilde{q}_i$, we obtain $\Omega = \sum_{i=1}^n tdp_i \wedge d\tilde{q}_i + \sum_{i=1}^n dp_i \wedge \tilde{q}_i dt$. Since the last term vanishes on the fibers, the form $\Omega_1 = t^{-1}\Omega = \sum_{i=1}^n dp_i \wedge d\tilde{q}_i$ is smooth, non-degenerate on the central fiber, and equal to $t^{-1}\Omega$ on the general fibers. This proves Theorem 1. ■

Closed 1-forms in a neighbourhood of a contractible Lagrangian variety

Proposition 2: Let (M, Ω) be a holomorphically symplectic manifold, and $X \subset M$ a compact holomorphic Lagrangian submanifold. Assume that M admits a proper, birational map which contracts X. **Then for any section η_0 of $\Omega^1 M|_X$ there exists a closed 1-form η of $\Omega^1 M$ defined in a neighbourhood of X such that $\eta|_X = \eta_0$.**

Proof. Step 1: Since X is rationally connected, the pullback map $\pi^*: H^0(\Omega^1 M) \longrightarrow H^0(\Omega^1 X) = 0$ vanishes. Therefore, η_0 vanishes on X, and we may consider η_0 as a section of the conormal bundle $N^*X \subset \Omega^1 M|_X$. We interpret sections of N^*X as 1-jets of functions on M constant on X. To find a closed form η such that $\eta|_X = \eta_0$, **it would suffice to find a function $f \in \mathcal{O}_M$ such that its 1-jet in the normal direction to X is equal to $\eta_0 \in H^0(N^*X)$.**

Step 2: Consider the exact sequence of sheaves

$$0 \longrightarrow (J_X)^2 \longrightarrow \mathcal{O}_M \xrightarrow{\delta} J^1 M|_X \longrightarrow 0$$

where δ takes a function and gives its 1-jet in X, and J_X is the ideal of X. To finish the proof of Proposition 2, **it would suffice to prove that δ is surjective on global sections.** This follows from Proposition 1, because $J^1 M|_X = \mathcal{O}_M/(J_X)^2$, and the natural map $H^0(\mathcal{O}_M) \longrightarrow H^0(\mathcal{O}_M/(J_X)^2)$ is surjective. ■

Mittag-Leffler systems

DEFINITION: Consider a diagram of sheaves $... \to H_i \to H_{i-1} \to ... \to H_0$, and let $H_{k,i}$ be the image of H_k in H_i. Clearly, $... \subset H_{k,i} \subset H_{k-1,i} \subset H_{k-2,i} \subset ....$ The diagram is called **a Mittag-Leffler system** if for each i, the sequence $H_{k,i} \supset H_{k+1,i} \supset ...$ stabilizes.

THEOREM: For any Mittag-Leffler system, the inverse limit commutes with the cohomology: $H^j(\varprojlim H_k) = \varprojlim H^j(H_k)$.

Proof: *Stacks Project, Lemma 10.86.4.* ∎

DEFINITION: Let $X \subset M$ be a complex subvariety, and $\mathcal{J} \subset \mathcal{O}_M$ an ideal sheaf. We say that the sheaf $\mathcal{J}$ **is supported in X** if for some $k > 0$ we have $J_X^k \subset \mathcal{J} \subset J_X$.

We change Proposition 1 to get more freedom in the choice of an ideal.

Surjectivity of restrictions

Proposition 1': Let (M, Ω) be a holomorphically symplectic manifold, and $X \subset M$ a compact holomorphic Lagrangian submanifold. Assume that M admits a proper, birational map to a Stein variety which contracts X. **Then the natural map $H^0(\mathcal{O}_M) \longrightarrow H^0(J^k)$ is surjective, for any $k \geqslant 0$, and some (and, therefore, any) ideal J supported in X.**

Proof. Step 1: Denote by $\hat{\mathcal{O}}_M$ and $\hat{J}_X$ the J_X-adic completions. Clearly, surjectivity of $H^0(\mathcal{O}_M) \longrightarrow H^0(J^k)$ implies the surjectivity of $H^0(\hat{\mathcal{O}}_M) \longrightarrow H^0(\hat{J}^k)$. **The converse implication follows from the Artin algebraization theorem,** which tells us that any formal solution of any system of equations over the adic completion of a ring can be chosen in its strict Hensel completion (the ring of germs of $H^0(\mathcal{O}_M)$ in X is clearly Henselian).

Step 2: From the exact sequence $0 \longrightarrow \hat{J}^k \longrightarrow \hat{\mathcal{O}}_M \longrightarrow \hat{\mathcal{O}}_M/\hat{J}^k \longrightarrow 0$ it follows that the surjectivity of $H^0(\hat{\mathcal{O}}_M) \longrightarrow H^0(\hat{J}^k)$ is implied by $H^1(\hat{J}^k) = 0$. Since J^i is a Mittag-Leffler system, **Proposition 2 would follow if we prove that $H^1(J^k/J^{k+1}) = 0$ for all k.**

Step 3: We deduced Proposition 1 from the following lemma.
Lemma 1: Let $X \subset M$ be a complex submanifold which admits a proper, birational map to a Stein variety which contracts X. **Then there exists an ideal J supported in X such that $H^i(J^k/J^{k-1}) = 0$, for any $i, k \geqslant 0$.**

Ample sheaves on Moishezon manifolds: results of Vo Van Tan

DEFINITION: A complex analytic space is called **1-convex** if it admits a proper holomorphic, bimeromorphic map to a Stein variety.

DEFINITION: Given a coherent sheaf A on X, denote by $L(A)$ the complex analytic space obtained as the relative spectrum of $\bigoplus \operatorname{Sym}^k(A)$ over X. **When A is a vector bundle, $L(A)$ is the total space A^*.**

DEFINITION: A coherent sheaf A on a compact complex analytic space X is called **ample** if for any coherent sheaf $\mathcal{F}$ on X there exists $k > 0$ such that $\operatorname{Sym}^k(A) \otimes \mathcal{F}$ is globally generated. It is called **cohomologically positive** if there exists $k > 0$ such that $H^i(\operatorname{Sym}^k(A) \otimes \mathcal{F}) = 0$ for all $i > 0$, and **weakly positive** if the zero section of $L(A)$ admits a 1-convex neighbourhood.

Ample sheaves on Moishezon manifolds: results of Vo Van Tan (2)

THEOREM: (Vo Van Tan)
These three conditions (ampleness, weak positivity, cohomological positivity) are equivalent.

Proof: *Vo Van Tan, On Grauert's conjecture and the characterization of Moishezon spaces, Commentarii Mathematici Helvetici volume 58, pages 678-686 (1983), Theorem 1.* ■

THEOREM: (Vo Van Tan)
Let $X \subset M$ be a bimeromorphically contractible subvariety, and J_X its ideal. **Then there exists an ample ideal sheaf $\mathcal{J}$ supported at X.**

Proof: *Vo Van Tan, On Grauert's conjecture and the characterization of Moishezon spaces, Commentarii Mathematici Helvetici volume 58, pages 678-686 (1983), Theorem 2.* ■

Lemma 1 immediately follows from this assertion, because a quotient of an ample sheaf is ample, and the cohomology of an ample sheaf on $\mathbb{C}P^n$ vanish by Kodaira.