C-symplectic structures and their applications: a new proof of Bogomolov's and Voisin's theorems.

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Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a vector space, and Symp an infinite-dimensional (Fréchet) manifold.

DEFINITION: Teichmüller space of symplectic structures on M is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$.

REMARK: Let $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ be the mapping class group of M. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, **is identified with the set of symplectic structures up to diffeomorphism.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Let M be compact. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

Proof of Moser's theorem: The period map $P: U \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion of infinite-dimensional smooth manifolds. By Moser's lemma, the fibers of P are 0-dimensional. **Therefore, P is locally a diffeomorphism.** ■

The proof of Moser's lemma

Moser's lemma: Let ω_t, $t \in [0,1]$ be a smooth family of symplectic structures on a compact manifold M. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then there exists a smooth family $\Psi_t \in \operatorname{Diff}_0(M)$ of diffeomorphisms such that $\Psi_t^* \omega_0 = \omega_t$.**

Proof: We construct Ψ_t as a solution of the equation $\Psi_t^{-1} \frac{d\Psi_t}{dt} = X_t$, where $X_t \in TM$ is a vector field depending on $t \in [0,1]$.

Step 1: Since all ω_t are cohomologous, the form $\frac{d\omega_t}{dt}$ is exact. This gives $\frac{d\omega_t}{dt} = d\eta_t$, where $\eta_t \in \Lambda^1(M)$ smoothly depends on $t \in [0,1]$. Let X_t be the vector field which satisfies $\omega_t \lrcorner X_t = \eta_t$. **Cartan's formula gives $\operatorname{Lie}_{X_t} \omega_t = d(\omega_t \lrcorner X_t) = d\eta_t = \frac{d\omega_t}{dt}$.**

Step 2: Let Ψ_t be the flow of diffeomorphisms obtained by integrating X_t. By construction, $\operatorname{Lie}_{X_t} \omega_t = \frac{d\omega_t}{dt}$. Integrating it in t, we obtain

$$\Psi_1^* \omega_0 = \int_0^1 \Psi_t \operatorname{Lie}_{X_t} \omega_t dt = \int_0^1 \frac{d\omega_t}{dt} dt = \omega_1.$$

■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

REMARK: The "usual definition": complex structure is an atlas on a manifold with differentials of all transition functions in $GL(n,\mathbb{C})$.

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

REMARK: An almost complex structure I **is uniquely determined by a subbundle $B \subset TM \otimes_{\mathbb{R}} \mathbb{C}$** such that $TM \otimes_{\mathbb{R}} \mathbb{C} = B \oplus \overline{B}$. Then we write $I = \sqrt{-1}$ on B and $I = -\sqrt{-1}$ on $\overline{B}$.

Holomorphically symplectic manifolds

DEFINITION: Let (M, I) be a complex manifold, and $\Omega \in \Lambda^2(M, \mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker \Omega \cap T_{\mathbb{R}}M = 0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega = 0$, and $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$.

REMARK: The equation $\Omega(IX, Y) = \sqrt{-1}\Omega(X, Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX, Y) = \sqrt{-1}\,\Omega(X, Y)$ and $I(Z) = -\sqrt{-1}\,Z$ for any $Z \in T^{0,1}(M)$, we have $\ker(\Omega) \supset T^{0,1}(M)$. Since $\ker \Omega \cap T_{\mathbb{R}}M = 0$, real dimension of its kernel is at most $\dim_{\mathbb{R}} M$, giving $\dim_{\mathbb{R}} \ker \Omega = \dim M$. **Therefore, $\ker(\Omega) = T^{0,1}M$.**

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M, I). **Then I is determined by Ω uniquely.**

C-symplectic structures

CLAIM: Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying** $\dim_{\mathbb{C}} T^{0,1}M = n$ **and** $T^{0,1}M \cap TM = 0$ (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\Big|_{T^{1,0}M} = \sqrt{-1}$ and $I\Big|_{T^{0,1}M} = -\sqrt{-1}$. ■

DEFINITION: Let M be a smooth $4n$-dimensional manifold. A closed complex-valued form Ω on M is called **C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and

$$T^{0,1}_\Omega(M) := \{v \in TM \otimes \mathbb{C} \ \mid \ \Omega \lrcorner v = 0\}.$$

Then $T^{0,1}_\Omega(M)$ satisfies assumptions of the claim above, hence **defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable.**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and it is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}}M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula. ■

Period map for holomorphically symplectic manifolds

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and Symp_S the space of all holomorphically symplectic forms. The quotient $\mathsf{Teich}_S := \frac{\mathsf{Symp}_S}{\mathsf{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\mathsf{Teich}_S \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

We want to prove that **the period map is locally an embedding.** This is immediately implied by the following version of Moser's lemma.

THEOREM: Let (M, I_t, Ω_t), $t \in [0, 1]$ be a family of holomorphic symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \mathsf{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Holomorphically symplectic Moser's lemma

THEOREM: Let (M, I_t, Ω_t), $t \in [0,1]$ be a family of holomorphic symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \operatorname{Diff}_0(M)$, such that $V_t^*\Omega_0 = \Omega_t$.**

Proof. Step 1: If we find a vector field X_t such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$, we have

$$V_{t_1}^*\Omega_0 = \int_0^{t_1} \operatorname{Lie}_{X_t}\Omega_t dt = \int_0^{t_1} \frac{d\Omega_t}{dt} dt = \Omega_{t_1}$$

where V_t is a diffeomorphism flow integrating X_t. **It remains to find $X_t \in T_{\mathbb{R}}M$.**

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb{R}} T_{\mathbb{R}}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective** (an exercise).

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Holomorphically symplectic Moser's lemma (2)

It remains to find $X_t \in T_{\mathbb R}M$ such that $\operatorname{Lie}_{X_t}\Omega_t = \frac{d}{dt}\Omega_t$**.**

Step 2: The contraction map $\Lambda^{2,0}M \otimes_{\mathbb R} T_{\mathbb R}M \longrightarrow \Lambda^{1,0}(M)$ **is surjective**.

Step 3: Since $\frac{d}{dt}\Omega_t$ is exact, one has $\frac{d}{dt}\Omega_t = d\alpha_t$. If α_t has Hodge type (1,0), we could obtain it as $\Omega_t \lrcorner X_t$ (Step 2), which gives $\frac{d}{dt}\Omega_t = d\alpha_t = d(\Omega_t \lrcorner X_t) = \operatorname{Lie}_{X_t}\Omega_t$. **It remains to find $\alpha_t \in \Lambda^{1,0}(M, I_t)$ such that $\frac{d}{dt}\Omega_t = d\alpha_t$.**

Step 4: Let $\Omega_t' := \frac{d}{dt}\Omega_t$ and $\dim_{\mathbb C} M = 2n$. Differentiating $\Omega_t^{n+1} = 0$ in t, we obtain $\Omega_t' \wedge \Omega_t^n = 0$. Since $\Phi := \Omega_t^n$ is a holomorphic volume form, the multiplication map $\Lambda^{0,2}(M) \xrightarrow{\wedge\Phi} \Lambda^{2n,2}(M)$ is an isomorphism of vector bundles. **Then $\Omega_t' \wedge \Omega_t^n = 0$ implies that $\Omega_t' \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$.**

Step 5: Using Step 3 and Step 4, we obtain that holomorphic Moser's lemma **is implied by the following statement.**

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Holomorphically symplectic Moser's lemma (3)

LEMMA: Let M be a complex manifold which satisfies $H^{0,1}(M) = 0$, and $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$ an exact form. **Then $\eta = d\alpha$, for some $\alpha \in \Lambda^{1,0}(M)$.**

Proof. Step 1: Let $\eta = d\beta$, where $\beta = \beta^{1,0} + \beta^{0,1}$. Since $\eta \in \Lambda^{1,1}(M) + \Lambda^{2,0}(M)$, we have $\overline{\partial}(\beta^{0,1}) = 0$. The first cohomology of the complex $(\Lambda^{0,*}(M), \overline{\partial})$ vanish, because $H^{0,1}(M) = 0$, **hence $\beta^{0,1} = \overline{\partial}\psi$, for some $\psi \in C^\infty M$.**

Step 2: This gives $\eta = d(\beta - d\psi)$, hence $\alpha := \beta - d\psi = \beta^{1,0} + \beta^{0,1} - \partial\psi - \beta^{0,1}$ **is a (1,0)-form which satisfies $\eta = d\alpha$.** ■

COROLLARY: Let CSymp be the space of all C-symplectic structures with C^∞-topology. Denote by Teich_C the corresponding Teichmüller space, $\operatorname{Teich}_C := \frac{\operatorname{CSymp}}{\operatorname{Diff}_0(M)}$. Define **the period map** $\operatorname{Per}: \operatorname{Teich}_C \longrightarrow H^2(M, \mathbb{C})$ mapping Ω to its cohomology class. **Then Per is locally a homeomorphism to its image.**

Proof: All fibers of Per are 0-dimensional. ■

Local Torelli theorem

Bogomolov's local Torelli theorem can be stated as follows.

Theorem 1: Consider a holomorphically symplectic manifold (M, Ω_0). Assume that the Hodge-de Rham spectral sequence degenerates in $H^2(M_{\Omega_0})$ and in $H^1(M_{\Omega_0})$. Assume, moreover, that $H^{2,0}(M_{\Omega_0})$ is generated by Ω_0. Let $U \subset$ CSymp be a small neighbourhood of $[\Omega_0] \in$ CSymp, and Per : $U \longrightarrow H^2(M, \mathbb{C})$ the period map. **Then Per is a local homeomorphism to the subset $\{\eta \in H^2(M, \mathbb{C}) \ \ | \ \ \eta^{n+1} = 0\}$.**

Proof: Local injectivity of the period map follows from Moser isotopy. It remains to prove the surjectivity.

Let Ω_t, $t \in]-\varepsilon, \varepsilon[$ be a deformation of a C-symplectic form, and $\Omega_t' := \frac{d\Omega_t}{dt}$. Differentiating $\Omega_t^{n+1} = 0$, we obtain $(n+1)\Omega_t' \wedge \Omega_t^n = 0$, which is equivalent to $\Omega_t' \in \Lambda^{2,0}(M_{\Omega_t}) \oplus \Lambda^{1,1}(M_{\Omega_t})$.

Conversely, whenever Ω_t' satisfies $\Omega_t' \in \Lambda^{2,0}(M_{\Omega_t}) \oplus \Lambda^{1,1}(M_{\Omega_t})$, we have $\frac{d\Omega_t^{n+1}}{dt} = 0$, which gives $\Omega_t^{n+1} = 0$ for all t.

Therefore, **the surjectivity of the period map is implied if we prove the following result.**

Surjectivity of the period map

PROPOSITION: In assumptions of Theorem 1, let Ω_0 be a C-symplectic structure, $[\eta_t] \in H^2(M, \mathbb{C})$ a family of cohomology classes, $t \in]-\varepsilon, \varepsilon[$, and $[\Omega_t] := [\Omega_0] + \int_0^t [\eta_t] dt$. Assume that $[\eta_t] \wedge [\Omega_t^n] = 0$ for all t. **Then, after shrinking the interval $]-\varepsilon, \varepsilon[$ if necessary, $[\Omega_t]$ can be represented by a C-symplectic form Ω_t in such a way that $\eta_t := \frac{\Omega_t}{dt}$ is cohomologous to $[\eta_t]$.**

Proof: Small deformations of a manifold with degenerate Hodge-de Rham spectral sequence also have degenerate Hodge-de Rham, by semi-continuity of $H^{p,q}(M)$. Represent Ω_t as a function of $t \in]-\varepsilon, \varepsilon[$, and let η_t be a (2,0)+(1,1) form on M_{Ω_t} representing $[\eta_t]$. Then Ω_t is a solution of the following non-linear differential equation

$$\frac{d\Omega_t}{dt} = \eta_t.$$

This equation is non-linear, because the choice of the form η_t depends on Ω_t; however, this ODE can be solved for small values of t. ■

Lagrangian submanifolds and C-symplectic structures

DEFINITION: Let (M,Ω) be a holomorphicalllly symplectic manifold, and $X\subset M$ a complex analytic subvariety. It is called **holomorphic Lagrangian** if $\dim X = \frac{1}{2}\dim M$, and $\Omega|_X = 0$ in all smooth points of X.

PROPOSITION: Let Ω be a C-symplectic form on M, and $X\subset M$ a submanifold, $\dim X = \frac{1}{2}\dim M$, such that $\Omega|_X = 0$. **Then X is holomorphic Lagrangian with respect to the complex structure induced by Ω.**

Proof: Write $\Omega = \omega_1 + \sqrt{-1}\,\omega_2$, where ω_1, ω_2 are real forms. The complex structure on M can be written as $I = \omega_1\circ\omega_2^{-1}$. However, ω_i map TX to the space of 1-forms vanishing on TX, hence $\omega_1\circ\omega_2^{-1}$ map TX to itself. ■

We proof Voisin's theorem on deformation of Lagrangian submanifolds.

THEOREM: (Voisin theorem). In assumptions of Theorem 1, let $\Omega_t, t\in]-\varepsilon,\varepsilon[$ be a family of C-symplectic structures, and $X\subset M_{\Omega_0}$ a holomorphic Lagrangian submanifold which satisfies dd^c-lemma in $\Lambda^2(X)$. Assume that the restriction $\Omega_t|_X$ is exact. **Then $X_0 = X\subset M_{\Omega_0}$ can be extended to a continuous family $X_t\subset M_{\Omega_t}$ of holomorphic Lagrangian submanifolds.**

A proof of Voisin's theorem

THEOREM: (Voisin theorem). In assumptions of Theorem 1, let $\Omega_t, t \in]-\varepsilon, \varepsilon[$ be a family of C-symplectic structures, and $X \subset M_{\Omega_0}$ a holomorphic Lagrangian submanifold which satisfies dd^c-lemma in $\Lambda^2(X)$. Assume that the restriction $\Omega_t|_X$ is exact. **Then $X_0 = X \subset M_{\Omega_0}$ can be extended to a continuous family $X_t \subset M_{\Omega_t}$ of holomorphic Lagrangian submanifolds.**

Proof. Step 1: Let $[\eta_t]$ be the cohomology class of $\frac{d\Omega_t}{dt}$. Since the family Ω_t is uniquely determined by its image under the period map, and this image is uniquely determined by the family $[\eta_t]$, **it would suffice to find a family Θ_t of C-symplectic structures such that $\Theta_0 = \Omega_0$, the derivative $\frac{d\Theta_t}{dt}$ is cohomologous to η_t, and $\Theta_t\Big|_{X_0} = 0$.**

Step 2: Let $\eta_t \in \Lambda^{2,0}(M_{\Theta_t}) + \Lambda^{1,1}(M_{\Theta_t})$ be a closed representative of the class $[\eta_t]$ for the complex manifold M_{Θ_t}. We find Θ_t as a solution of an equation $\frac{d\Theta_t}{dt} = \eta_t - d\alpha_t$, where $\eta_t|_X = d\alpha_t|_X$. **Then Voisin Theorem follows from $\Theta_0|_X = 0$ and $\frac{d\Theta_t}{dt}|_X = 0$.**

A proof of Voisin's theorem (2)

Step 2: Let $\eta_t \in \Lambda^{2,0}(M_{\Theta_t}) + \Lambda^{1,1}(M_{\Theta_t})$ be a closed representative of the class $[\eta_t]$ for the complex manifold M_{Θ_t}. We find Θ_t as a solution of an equation $\frac{d\Theta_t}{dt} = \eta_t - d\alpha_t$, where $\eta_t|_X = d\alpha_t|_X$, where $X = X_0$. **Then Voisin Theorem follows from $\Theta_0|_X = 0$ and $\frac{d\Theta_t}{dt}|_X = 0$.**

Step 3: Since $\eta_t|_X$ is exact, we can write $\eta_t|_X = d\beta_t$, for a smooth family β_t of 1-forms on X. Since $\eta_t \in \Lambda^{2,0}(M_{\Theta_t}) + \Lambda^{1,1}(M_{\Theta_t})$, we have $\overline{\partial}_t(\beta_t^{0,1}) = 0$, where $\overline{\partial}_t$ means the $\overline{\partial}$-operator taken on M_{Θ_t}.

Since X_{Ω_t} is Kähler for small t, the dd^c-lemma implies that there exists $f \in C^\infty X$ such that $d\beta_t^{0,1} = \partial_t\beta_t^{0,1} = \partial_t\overline{\partial}_t f = -d(\partial_t f)$. Then $d\beta = d\beta^{1,0} - d(\partial_t f)$. **Replacing β by $\beta^{1,0} - \partial_t f$, we may assume that $\eta_t|_X = d\beta_t$, where β_t is a (1,0)-form.**

Step 4: We extend β_t to a smooth family α_t of (1,0)-forms on M. Then $\eta_t - d\alpha_t$ is a family of $(2,0)+(1,1)$-forms on M_{Θ_t}, vanishing on X. **We have constructed a family Θ_t of C-symplectic structures vanishing on X, and equivalent to Ω_t.** ■