Hyperholomorphic sheaves

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The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

DEFINITION: $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

REMARK: For any C^∞-bundle $(B, \overline{\partial})$ equipped with a holomorphic structure operator, **the sheaf $\mathcal{B} := \ker \overline{\partial}$ is a holomorphic bundle, and $B = \mathcal{B} \otimes_{\mathcal{O}_M} C^\infty M$.** This means that **the category of holomorphic bundles is equivalent to the category of C^∞-bundles equipped with a holomorphic structure operator.**

Connections and holomorphic structure operators

DEFINITION: let (B,∇) be a smooth bundle with connection and a holomorphic structure $\bar\partial\ B \longrightarrow \Lambda^{0,1}(M)\otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ B \longrightarrow \Lambda^{0,1}(M)\otimes B, \quad \nabla^{1,0}:\ B \longrightarrow \Lambda^{1,0}(M)\otimes B.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMARK: The curvature of a Chern connection on B is an End(B)-valued (1,1)-form: $\Theta_B \in \Lambda^{1,1}(\mathrm{End}(B))$.

REMARK: A converse is true, too. Given a Hermitian connection ∇ on a vector bundle B with curvature in $\Lambda^{1,1}(\mathrm{End}(B))$, we obtain a holomorphic structure operator $\bar\partial = \nabla^{0,1}$. Then, **$\nabla$ is a Chern connection of $(B,\bar\partial)$.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the Levi-Civita connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

THEOREM: (Calabi-Yau)
A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I\in \mathbb{C}P^1} \Lambda^{1,1}_I(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: Let M be a compact hyperkähler manifold. Then $SU(2)$ preserves harmonic forms, hence **acts on cohomology.**

CLAIM: All Chern classes of hyperholomorphic bundles are $SU(2)$-invariant.

Proof: Use $\Lambda^{2p}(M)_{SU(2)} = \bigcap_{I\in \mathbb{C}P^1} \Lambda^{p,p}_I(M)$. ■

REMARK: **Converse is also true** (for stable bundles). See the slide 7.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\text{slope}(F) := \frac{1}{\text{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\text{slope}(F') < \text{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \text{slope}(F) \cdot \text{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: **Yang-Mills connections minimize the integral**

$$\int_M |\Theta_B|^2 \text{Vol}_M$$

Kobayashi-Hitchin correspondence: (Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Kobayashi-Hitchin correspondence and hyperholomorphic bundles

CLAIM: Let M be a hyperkähler manifold. Then for any $SU(2)$-invariant 2-form $\eta \in \Lambda^2(M)$, one has $\eta \wedge \omega^{n-1} = 0$.

COROLLARY: **Any bundle admitting hyperholomorphic connection is Yang-Mills**, of slope 0 (and hence polystable).

REMARK: This implies that **a hyperholomorphic connection on a given holomorphic vector bundle is unique** (if exists). Such a bundle is called **hyperholomorphic**.

THEOREM: Let B be a polystable holomorphic bundle on (M, I), where (M, I, J, K) is hyperkähler. Then the (unique) **Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.**

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$-invariant $c_1(B)$ and $c_2(B)$ **is a hyperkähler variety** (possibly singular).

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure (that is, a complex structure outside of a certain countable set). **Then any stable bundle on (M, L) is hyperholomorphic.**

Calibrations

DEFINITION: (Harvey-Lawson, 1982)
Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

DEFINITION: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

DEFINITION: A **calibration** is a precalibration which is closed.

DEFINITION: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by η** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

Calibrations (2)

REMARK: Clearly, for any precalibration η, one has

$$\mathrm{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\mathrm{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, the inequality (*) implies that Z minimizes the Riemannian volume in its homology class.

DEFINITION: A subvariety Z is called **minimal** if for any sufficiently small deformation Z' of Z in class C^1, one has $\mathrm{Vol}(Z') \geqslant \mathrm{Vol}(Z)$.

REMARK: Calibrated subvarieties are obviously minimal.

EXAMPLE: (Wirtenger's inequality)
Let ω be a Kähler form. **Then $\frac{\omega^d}{d!2^d}$ is a calibration which calibrates d-dimensional complex subvarieties.** In patricular, **complex subvarieties in Kähler manifolds are minimal**.

REMARK: In most applications, **calibrations are parallel with respect to the Levi-Civita form.** To simplify the exposition, we tacitly assume that **all calibrations we consider are parallel.**

Calibrated instantons

DEFINITION: Let M be n-dimensional Riemannian manifold, and $\Phi \in \Lambda^{n-4}(M)$ a calibration. Consider the map $R: \Lambda^2(M) \longrightarrow \Lambda^2(M)$ taking η to $*(\eta \wedge \Phi)$. **Calibrated instanton** is a vector bundle (B, ∇) with orthogonal connection over M with curvature $\Theta_B \in \Lambda^2(M) \otimes \operatorname{End}(B)$ which satisfies $R(\Theta_B) = \Theta_B$.

REMARK: This theory was developed by Donaldson, Thomas, Tian, Tao, and has same properties as the usual instanton equation: Uhlenbeck compactness, ellipticity, deformation theory and so on.

EXAMPLE: Let $\dim M = 4$, $\Phi = 1$. In this case, $R^2 = 1$ and $R = *$ has two eigenspaces, $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$. The calibrated instantons in this case are and **selfdual connections** with $\Theta_B \in \Lambda^+ \otimes \operatorname{End}(B)$.

EXAMPLE: (M, ω) is Kähler, $\Phi = \frac{\omega^2}{4}$. The instantons corresponding to the negative eigenspace are called **Yang-Mills bundles**, and **the corresponding category is equivalent to the category of (poly-)stable holomorphic bundles on (M, ω).**

Calibrations on hyperkähler manifolds

THEOREM: (Grantcharov, V.)
Let (M, I, J, K, g) be a hyperkähler manifold, $\omega_I, \omega_J, \omega_K$ the corresponding symplectic forms, and $\Theta_p := \frac{(\omega_I^2+\omega_J^2+\omega_K^2)^p}{c_p}$ the standard $SU(2)$-invariant $4p$-form normalized by $c_p = \sum_{k=1}^p \frac{(p!)^2}{(k!)^2}(2k)!4^{p-k}$. **Then Θ_p is a calibration, and its faces are p-dimensional quaternionic subspaces of TM.**

REMARK: The corresponding calibrated subvarieties are called **trianalytic subvarieties**. These are subvarieties which are complex analytic with respect to I, J and K.

THEOREM: In the above assumptions, let $\dim_{\mathbb{R}} M = 4n$, and $\Theta_{n-1} \in \Lambda^{4n-4}(M)$ be the calibration defined above. Then the corresponding operator $R(\eta) = *(\eta \wedge \Theta_{n-1})$ has two eigenspaces on $\Lambda^2(M)$, the space $\Lambda^2(M)^{SU(2)}$ of $SU(2)$-invariant forms and the space of 2-forms of weight 2 with respect to $SU(2)$. **The instantons associated with $\Lambda^2(M)^{SU(2)}$ are precisely the hyperholomorphic bundles.**

Bando and Siu: admissible connections with singularities

DEFINITION: Let M be Kähler and $Z \subset M$ a closed subset of Hausdorff codimension $\geqslant 4$. A Chern connection ∇ on a bundle B is called **admissible** if the form $\operatorname{Tr}(\Theta_B \wedge \Theta_B)$ is integrable, and $|\operatorname{Tr}\Theta_B|$ is bounded.

THEOREM: (Bando, Siu)
The bundle B on $M\backslash Z$ **can be extended to a coherent sheaf on M if and only if it admits an admissible connection.**

THEOREM: (Bando, Siu)
A torsion-free coherent sheaf on M **is polystable if and only if its non-singular part admits an admissible connection with $\Lambda\Theta_B = const\, \mathrm{Id}_B$.**

DEFINITION: A **polystable reflexive hyperholomorphic sheaf** on a hyperkahler manifold is a direct sum of stable reflexive sheaves with $c_1(B), c_2(B)$ $SU(2)$-invariant.

THEOREM: A polystable reflexive hyperholomorphic sheaf F **admits an admissible connection with $SU(2)$-invariant curvature.** Conversely, **if F is reflexive and admits an admissible connection with $SU(2)$-invariant curvature, it is hyperholomorphic.**

Instantons without a bound on $\operatorname{Tr}(\Theta_B \wedge \Theta_B)$

EXAMPLE: Suppose $\pi: M_1 \longrightarrow M_2$ be a map of compact hyperkähler manifolds which is submersive and holomorphic in I, J, K, and B a hyperholomorphic bundle on M_1. Then the higher cohomology sheaves $R^i\pi_*(B)$ are hyperholomorphic outside of their singularities. **If the corresponding connection is admissible, this would imply that $R^i\pi_*(B)$ is hyperholomorphic.**

QUESTION: Is it always admissible? Yes if the singular set has dimension 0, otherwise unknown. The best we can do with singularities is to show that they have complex codimension 3.

QUESTION: Is the sheaf $R^i\pi_*(B)$ always hyperholomorphic? The answer is "yes", but the argument is not straightforward.

Hyperholomorphic sheaves and stability

QUESTION: Let M be compact hyperkähler, and F a reflexive coherent sheaf on (M,I). ssume that F is equipped with a Hermitian connection ∇ defined outside of the singular set of F, and the curvature of ∇ is $SU(2)$-invariant. **Can we prove that F is hyperholomorphic?**

First of all, since the first two Chern classes of F are of type $(1,1)$ and $(2,2)$ with respect to all complex structures induced by quaternions, **the (poly-)stability would imply that B is holomorphic.**

This is what I prove, using analytic estimates. However, **the hyperholomorphic connection on F is not necessarily ∇! It is obtained using the Kobayashi-Hitchin correspondence.**

THEOREM: Let M be a compact hyperkähler manifold, I an induced complex structire, and F a reflexive sheaf on (M,I) which cannot be decomposed onto a direct sum of non-trivial coherent sheaves. Assume that F is equipped with a Hermitian connection ∇ defined outside of the singular set of F, and the curvature of ∇ is $SU(2)$-invariant. **Then F is stable.**