Hyperholomorphic bundles on hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the Levi-Civita connection preserves I, J, K.

REMARK: A hyperkähler manifold is **holomorphically symplectic:** the 2-form $\omega_J + \sqrt{-1}\,\omega_K$ **is holomorphic on** (M, I).

Converse is also true:

THEOREM: (Calabi-Yau)
A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

REMARK: In many cases, **"hyperkähler manifold" means "holomorphically symplectic complex manifold of Kähler type".**

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ B \longrightarrow \Lambda^{0,1}(M) \otimes B, \quad \nabla^{1,0}:\ B \longrightarrow \Lambda^{1,0}(M) \otimes B.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMARK: The curvature of a Chern connection on B is an End(B)-valued (1,1)-form: $\Theta_B \in \Lambda^{1,1}(\mathrm{End}(B))$.

REMARK: A converse is true, too. Given a Hermitian connection ∇ on a vector bundle B with curvature in $\Lambda^{1,1}(\mathrm{End}(B))$, we obtain a holomorphic structure operator $\overline{\partial} = \nabla^{0,1}$. Then, **$\nabla$ is a Chern connection of $(B, \overline{\partial})$.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2) = U(1, \mathbb{H})$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: The hyperholomorphic connections induce holomorphic structure on B over (M, I), (M, J), (M, K) and over (M, L) for any quaterion $L \in \mathbb{H}$ such that $L^2 = -1$.

$SU(2)$-action on cohomology

REMARK: All manifolds **will be tacitly assumed compact.**
DEFINITION: Let $H^m(M) = \bigoplus_{p+q=m} H^{p,q}(M,I)$ be cohomology of a Kähler manifold (M,I). We represent $U(1)$ as the unit circle in $\mathbb{C}$ **The complex rotation**, or **Hodge rotation** is $U(1)$-action $\rho_I(t)$ on $H^m(M,\mathbb{R})$ is $\rho_I(t)(\alpha) = t^{p-q}\alpha$ for $\alpha \in H^{p,q}(M)$.

PROPOSITION: Let M be a hyperkähler manifold, and $G \subset \operatorname{Aut}(H^*(M,\mathbb{R}))$ be group generated by the Hodge rotations for I, J, K. **Then G is naturally isomorphic to a quotient of $SU(2) = U(1,\mathbb{H})$.**

One of the reasons for this observation is the following theorem. Define $\mathfrak{sp}(1,1)$ as the Lie algebra of quaternionic-linear transforms of $V = \mathbb{H}^2$ preserving a quaternionic pseudo-Hermitian form of signature (1,1).

THEOREM: Let $\omega_I, \omega_J, \omega_K$ be the Kähler forms on a hyperkähler manifold, and and $L_I, \Lambda_I, L_J, \Lambda_J, L_K, \Lambda_K$ the corresponding generators of the Lefschetz $\mathfrak{sl}(2)$-triples. Then the Lie algebra $\mathfrak{a}$ generated by the action of $L_I, \Lambda_I, L_J, \Lambda_J, L_K, \Lambda_K$ on $\Lambda^*(V)$ **is isomorphic to $\mathfrak{sp}(1,1) \cong \mathfrak{so}(1,4)$.** Moreover, its Lie group **contains the Hodge rotations associated with I, J, K.**

REMARK: The group $SU(2)$ generated by Hodge rotations acts on differential forms, and **this action commutes with the Laplacian.**

Hyperholomorphic bundles

DEFINITION: Let F be a coherent sheaf on a Kähler manifold (M,I,ω). **The degree** of F is $\int_M c_1(F)\wedge\omega^{n-1}$

DEFINITION: Let F be a coherent sheaf over an n-dimensional Kähler manifold (M,ω), and $\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$. A torsion-free sheaf F is called **stable** if for all subsheaves $F'\subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: Let B be a stable bundle on a manifold (M,I,ω) equipped with a hyperkähler structure (I,J,K). It is called **hyperholomorphic** if $c_1(B)$ and $c_2(B)$ is $SU(2)$-invariant. It is called **projectively hyperholomorphic** if $c_2(\operatorname{End} B)$ is $SU(2)$-invariant.

THEOREM: A bundle B is hyperholomorphic **if and only if it admits a Chern conection with its curvature form $\Theta_B\in\Lambda^2(M)\otimes\operatorname{End}(B)$ $SU(2)$-invariant,** and such connection is unique.

COROLLARY: Let $L = aI+bJ+cK$ be a unit quaternion, $L^2=-1$; we use the same letter to denote a complex structure $aI+bJ+cK$ on M. Then **there is a bijective correspondence between the hyperholomorphic bundles on (M,I) and on (M,L).**

Deforming the complex structure

DEFINITION: A hyperkähler manifold (M, I, J, K) is called **of maximal holonomy**, or **IHS** if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

THEOREM: Let L be a general complex structure of hyperkähler type on a hyperkähler manifold M of maximal holonomy. Then **all stable bundles on (M, L) are hyperholomorphic for any hyperkähler structure on (M, L).**

THEOREM: Let (M, I, J, K) be a maximal holonomy hyperkähler manifold, B a hyperholomorphic bundle, and $W \subset H^2(M, \mathbb{R})$ the smallest rational subspace which contains the Kähler classes $\omega_I, \omega_J, \omega_K$. Let I_1 be a complex deformation of I of hyperkähler type such that the corresponding holomorphic symplectic form Ω_{I_1} satisfies $[\Omega_{I_1}] \in W \otimes_{\mathbb{R}} \mathbb{C}$. Choose a hyperkähler structure (I_1, J_1, K_1) such that $\omega_{I_1}, \omega_{J_1}, \omega_{K_1} \in W$. **Then B can be deformed to a hyperholomorphic bundle on B_1 on (M, I_1, J_1, K_1).** Moreover, **there is a natural diffeomorphism** from the moduli space of deformations of B to the moduli of deformations of B_1.

Idea of the proof: Use the twistor rotations for different hyperkähler structures (I', J', K') with $\omega_{I'}, \omega_{J'}, \omega_{K'} \in W$ to connect I to I_1.

Deformation theory for stable bundles

DEFINITION: Let X be a Kähler manifold, and B a stable bundle on X. A space $\mathcal{M}$ is **a (coarse) moduli space of deformations of B** if its points are in bijective correspondence with isomorphism classes of stable bundles which are deformationally equivalent to B, and for any deformation $\mathcal{B}$ of B over $Y \times X$, there exists a unique morphism $\varphi : Y \longrightarrow \mathcal{M}$ such that for all $s \in Y$, the restriction of $\mathcal{B}$ to $\{s\} \times X$ is isomorphic to the bundle on X associated with the point $\varphi(s) \in \mathcal{M}$.

CLAIM: Let B be a stable bundle on a Kähler manifold (X, I), and $\mathcal{M}$ its moduli space. Then locally around B **the space $\mathcal{M}$ is naturally embedded in the vector space $H^1(X, \operatorname{End}(B))$;** the image is given by a sequence of homogeneous equations of degree $2, 3, ...$, called **obstructions.**
CLAIM: The first of these obstructions is the Yoneda square map $v \to v^2$ taking $v \in H^1(X, \operatorname{End}(B)) = \operatorname{Ext}^1(B, B)$ to $v^2 \in H^2(X, \operatorname{End}(B)) = \operatorname{Ext}^2(B, B)$.

DEFINITION: We say that the deformation of a stable bundle B **has only quadratic obstructions** if all higher obstructions vanish, and the image of $\mathcal{M}$ in $H^1(X, \operatorname{End}(B))$ is given by the equation $v^2 = 0$.

THEOREM: Let (M, I, J, K) be a hyperkähler manifold, and B a stable projectively hyperholomorphic bundle on (M, I). Then its deformation **has only quadratic obstructions**.

Trianalytic subvarieties

DEFINITION: A complex structure $L = aI + bJ + cB$, with $a^2 + b^2 + c^2 = 1$, is called **induced complex structure**, or **induced by the quaternion action**.

DEFINITION: Let (M, I, J, K, g) be a hyperkähler manifold. A complex subvariety $Z \subset (M, I)$ is called **trianalytic** if it is complex analytic with respect to J and K.

CLAIM: A trianalytic subvariety $Z \subset (M, I)$ **is complex analytic with respect to any induced complex structure $L = aI + bJ + cB$**.

THEOREM: Let $Z \subset (M, L)$ be a complex subvariety in (M, L) for general induced $L = aI + bJ + cB$. **Then Z is trianalytic.**

REMARK: There are trianalytic subvarieties in any deformation of a generalized Kummer manifold. However, **a general deformation of a Hilbert scheme of K3 has no positive-dimensional subvarieties**. We don't know about O'Grady's examples.

Almost complex structures on real analytic varieties

DEFINITION: Let I be an ideal sheaf in the ring of real analytic functions in an open ball B in $\mathbb{R}^n$. The set of common zeroes of I is equipped with a structure of ringed space, with $\mathcal{O}(B)/I$ as the structure sheaf. We denote this ringed space by $Spec(\mathcal{O}(B)/I)$. A **real analytic variety** is a ringed space which is locally isomorphic to $Spec(\mathcal{O}(B)/I)$, for some ideal $I \subset \mathcal{O}(B)$, such that the natural sheaf morphism $\mathcal{O}(X) \longrightarrow C(X)$ is injective.

REMARK: This map does not need to be injective, even when X is a real analytic space underlying a complex variety.

DEFINITION: An almost complex structure on a real analytic variety M is an endomorphism $I: \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$ satisfying $I^2 = -\mathrm{Id}$.

DEFINITION: Let X be a complex analytic variety. The **real analytic variety underlying X** (denoted by $X^r_{\mathbb{R}}$) is a ringed space with the same topology as X, but with a different structure sheaf, denoted $\mathcal{O}_{X^r_{\mathbb{R}}}$. The sheaf $\mathcal{O}_{X^r_{\mathbb{R}}}$ is obtained as the image of the natural map from the sheaf of the real analytic functions on X in $C(X)$.

THEOREM: A real analytic variety underlying a given complex variety **is equipped with a natural almost complex structure.** In this case, this almost complex structure is called **integrable**.

Deligne-Simpson singular hyperkähler varieties

Deligne and Simpson defined singular hyperkähler varieties in terms of twistor spaces. I will give an equivalent definition, without mention twistors. **This definition is not equivalent to the modern definition, which is due to Beauville.**

DEFINITION: Let M be a real analytic variety equipped with almost complex structures I, J and K, such that $I \circ J = -J \circ I = K$. Then M is called **an almost hypercomplex variety.** We say that (M, I, J, K) is a **singular hypercomplex variety** if (M, I) and (M, J) is integrable.
CLAIM: In this case, an almost complex structure $L = aI + bJ + cK$ **is integrable for any unit quaternion L such that $L^2 = -1$.**

A caution: Take the quotient M/G of a hypercomplex manifold by an action of a finite group G, preserving the hypercomplex structure. **Then M/G is** *not* **hypercomplex, unless G acts freely.**

EXAMPLE: Let $Z \subset M$ be a trianalytic subvariety in a hyperkähler manifold. **Then Z is singular hypercomplex.**

EXAMPLE: Let projectively hyperholomorphic bundle on a hyperkähler manifold, and $\mathcal{M}$ moduli space of deformations of B. **Then $\mathcal{M}$ is a hyperholomorphic variety.**

The desingularization theorem

THEOREM: Let M be a hypercomplex variety, and I an integrable induced complex structure. **Then the normalization of (M,I) is smooth.** Moreover, M is locally isomorphic to the variety $\bigcup_i V_i$, where $V_i \subset \mathbb{C}^{2n}$ is a collection of quaternionic subspaces in $\mathbb{C}^{2n} = \mathbb{H}^n$.

COROLLARY: Let Z be the moduli of projectively hyperholomorphic bundles, or a trianalytic subvariety in a hyperkähler manifold. **Then its normalization is hyperkähler**.

REMARK: If we could take care of compactness, **this could bring us to new examples of hyperkähler manifolds!**

REMARK: In the next slide, **I will say "has no subvarieties" meaning "has no positive-dimensional subvarieties".**

Manifolds without subvarieties

REMARK: A hyperkähler manifold (M, I, J, K) has no trianalytic subvarieties **if and only if (M, L) has no complex subvarieties for general induced complex structure $L = aI + bJ + cK$.**

THEOREM: Let (M, I, J, K) be a hyperkähler manifold with no trianalytic subvarieties, and B a projectively hyperholomorphic bundle. Assume that any semistable deformation of B is stable (this happens, for instance, when $\operatorname{rk} B = 2$ and $c_1(B)$ is not divisible by 2, or when $\operatorname{rk} B = 23$ and $c_1(B)$ is not divisible by 3). **Then the deformation space of B is compact, and its normalization is hyperkähler**.

Idea of the proof: The deformation space is compactified by coherent sheaves, which might only have isolated singularities because M has no trianalytic subvarieties. These isolated singularities can be resolved the same way as we resolve the singularities of singular hypercomplex varieties. The resolution gives a quaternionic Kähler instanton bundle over $\mathbb{C}P^{2n-1} = \operatorname{Tw}(\mathbb{H}P^n)$. This leads to a contradiction, because a smooth family of vector bundles needs to have constant Chern classes.

EXAMPLE: Since a general deformation of $K3^{[n]}$ has no subvarieties, **this theorem can be applied to hyperholomorphic bundles on $K3^{[n]}$.**

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\mathsf{slope}(F) := \frac{1}{\mathsf{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\mathsf{slope}(F') < \mathsf{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \mathsf{slope}(F) \cdot \mathsf{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \mathsf{Vol}_M$$

Kobayashi-Hitchin correspondence: (Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Kobayashi-Hitchin correspondence and hyperholomorphic bundles

CLAIM: Let M be a hyperkähler manifold. Then for any $SU(2)$-invariant 2-form $\eta \in \Lambda^2(M)$, one has $\eta \wedge \omega^{n-1} = 0$.

COROLLARY: **Any bundle admitting hyperholomorphic connection is Yang-Mills**, of slope 0 (and hence polystable).

REMARK: This implies that **a hyperholomorphic connection on a given holomorphic vector bundle is unique** (if exists). Such a bundle is called **hyperholomorphic**.

THEOREM: Let B be a polystable holomorphic bundle on (M, I), where (M, I, J, K) is hyperkähler. Then the (unique) **Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.**

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$-invariant $c_1(B)$ and $c_2(B)$ **is a hyperkähler variety** (possibly singular).

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure (that is, a complex structure outside of a certain countable set). **Then any stable bundle on (M, L) is hyperholomorphic.**

Bando and Siu: admissible connections with singularities

DEFINITION: Let M be Kähler and $Z \subset M$ a closed subset of Hausdorff codimension $\geqslant 4$. A Chern connection ∇ on a bundle B is called **admissible** if the form $\operatorname{Tr}(\Theta_B \wedge \Theta_B)$ is integrable, and $|\operatorname{Tr} \Theta_B|$ is bounded.

THEOREM: (Bando, Siu)
The bundle B on $M\backslash Z$ **can be extended to a coherent sheaf on M if and only if it admits an admissible connection.**

THEOREM: (Bando, Siu)
A torsion-free coherent sheaf on M **is polystable if and only if its non-singular part admits an admissible connection with $\Lambda\Theta_B = const \operatorname{Id}_B$.**

DEFINITION: A **polystable reflexive hyperholomorphic sheaf** on a hyperkahler manifold is a direct sum of stable reflexive sheaves with $c_1(B), c_2(B)$ $SU(2)$-invariant.

THEOREM: A polystable reflexive hyperholomorphic sheaf F **admits an admissible connection with $SU(2)$-invariant curvature.** Conversely, **if F is reflexive and admits an admissible connection with $SU(2)$-invariant curvature, it is hyperholomorphic.**