Hyperbolic geometry and the proof of Morrison-Kawamata cone conjecture (1)

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Complex Geometry: discussion meeting

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Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**. The set of all Kähler classes is called **Kähler cone**.

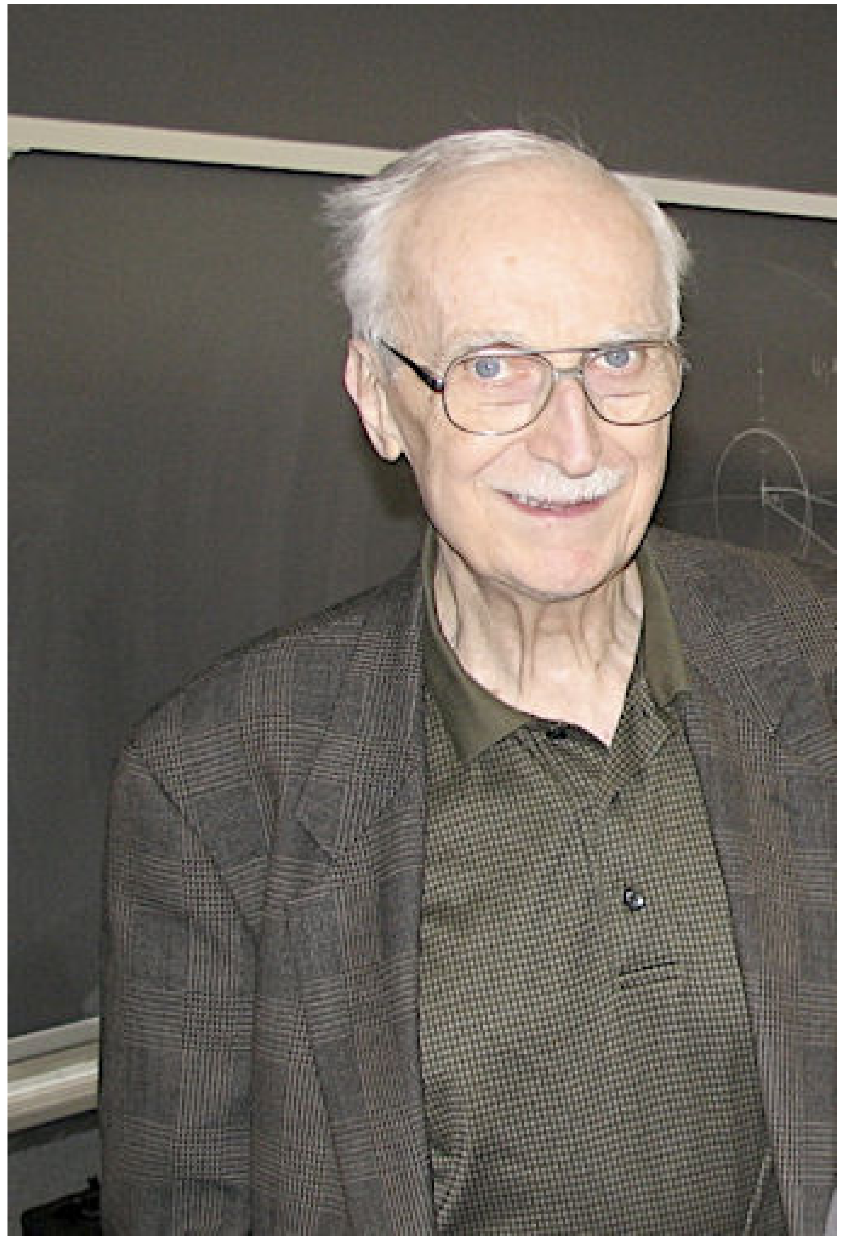
Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

CLAIM: A hyperkähler manifold (M,I,J,K) is **holomorphically symplectic** (equipped with a holomorphic, non-degenerate 2-form). Then M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on (M,I).** ■

THEOREM: (Calabi-Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.



Eugenio Calabi,
born 11 May 1923

Levi-Civita connection and Kähler geometry

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g)=0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

THEOREM: Let (M,I,g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) (M,I,g) **is Kähler**

(ii) One has $\nabla(I)=0$, where ∇ is the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla} : B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: Holonomy of a Riemannian manifold is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

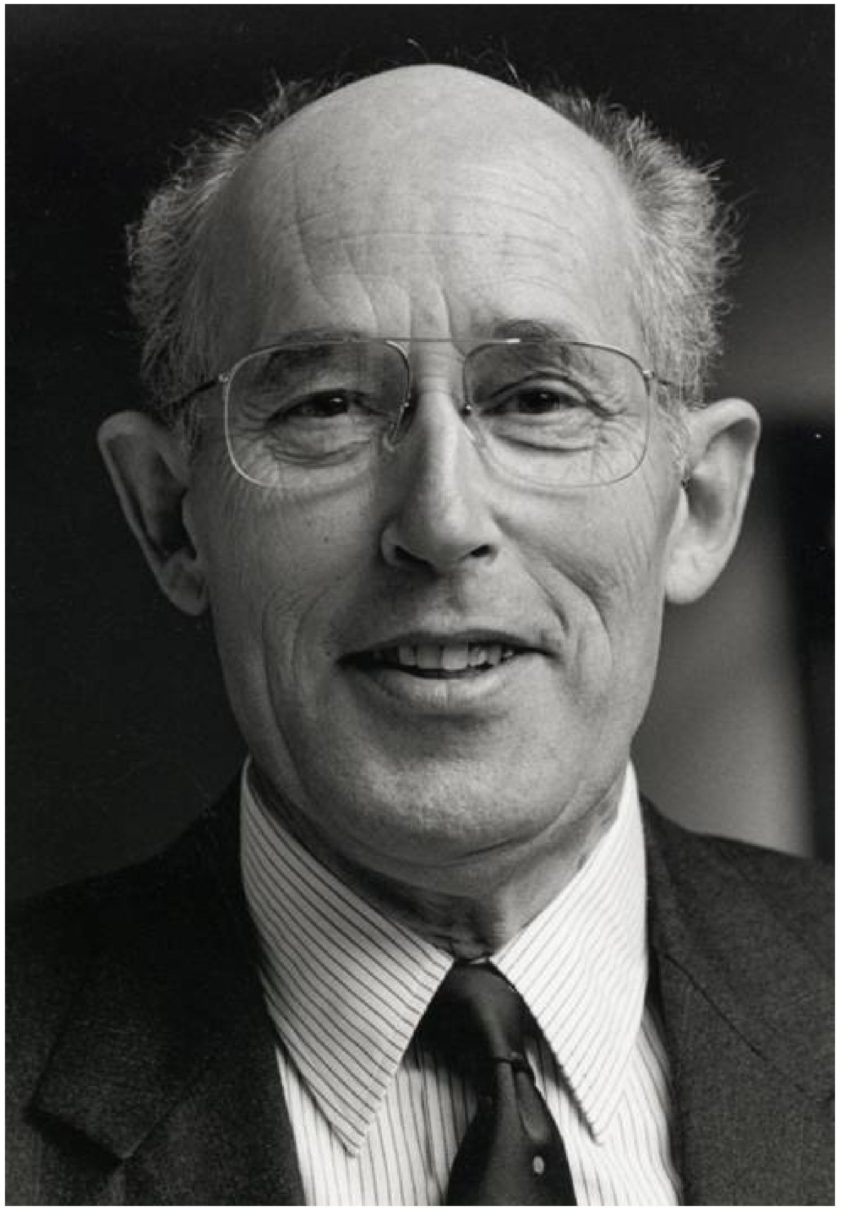
REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

The Berger's list

THEOREM: (de Rham)
A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger's classification of holonomies, 1955)
Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds



Marcel Berger,
14 April 1927 – 15 October 2016

Subject of these lectures

1. Determine the shape of the Kähler cone of a hyperkähler manifold. It turns out that it is determined by a quadratic inequality and a set of linear inequaities associated with the rational curves.

2. Interpret various quantities associated with a hyperkäher manifold, such as its automorphism group and its moduli space, im terms of the shape of the Kähler cone.

3. Associate a hyperbolic manifold to each hyperkähler manifold. Interpret the statement about the shape of the Kähler cone as a statement about this manifold.

4. Using ergodic theory and hyperbolic geometry, prove that the group of holomorphic automorphisms of a hyperkähler manifold acts on the polyhedral faces of its Kähler cone with finitely many orbits ("Morrison-Kawamata cone conjecture").

The results are obtained in a serie of joint papers with Ekaterina Amerik.

Calabi-Yau manifolds

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M) = \Omega^n M$. Denote by Θ_K the curvature of the connection on $K(M)$ induced by Levi-Civita connection. The **Ricci curvature** Ric of M is a symmetric 2-form $\operatorname{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

REMARK: Converse is also true: **any Ricci-flat Kähler manifold has a finite covering which is Calabi-Yau.** This is due to Bogomolov.

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

REMARK: Its proof is based on spinors: η gives a harmonic spinor, and **on a Ricci-flat Riemannian spin manifold, any harmonic spinor is parallel.**

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

Hyperkähler manifold

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$** (a group of complex unitary matrices preserving a complex-linear symplectic form).

DEFINITION: A holomorphically symplectic Kähler manifold with a Calabi-Yau metric is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

REMARK: Let M be a 2-dimensional complex manifold with holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

EXAMPLE: Take a 2-dimensional complex torus T, then all the singularities of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

K3 surfaces

DEFINITION: A K3-surface is a deformation of a Kummer surface.

"K3: Kummer, Kähler, Kodaira" (a name is due to A. Weil).

"Faichan Kangri (K3) is the 12th highest mountain on Earth."

THEOREM: Any complex compact surface with $c_1(M) = 1$ and $H^1(M) = 0$ **is isomorphic to K3.** Moreover, **it is hyperkähler.**

Hilbert schemes

REMARK: **A complex surface** is a 2-dimensional complex manifold.

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: **A Hilbert scheme of K3**.

EXAMPLE: Let T is a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds of maximal holonomy are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Ricci-flat Riemannian manifold with $\pi_1(M)$ infinite. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

COROLLARY: A fundamental group of a compact Ricci-flat Riemannian manifold is **"virtually polycyclic": it is projected to a free abelian subgroup with finite kernel.**

REMARK: This is equivalent to any compact Ricci-flat manifold having a finite covering which has free abelian fundamental group.

REMARK: This statement contains the Bieberbach's solution of Hilbert's 18-th problem on classification of crystallographic groups.

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}\mathrm{ol}(M_l) = Sp(n_l)$, $\mathcal{H}\mathrm{ol}(K_l) = SU(m_l)$

Holomorphic Euler characteristic

DEFINITION: A holomorphic Euler characteristic $\chi(M)$ of a Kähler manifold is a sum $\sum(-1)^p \dim H^{p,0}(M)$.

THEOREM: (Riemann-Roch-Hirzebruch) For an n-fold, $\chi(M)$ **can be expressed as a polynomial expressions of the Chern classes,** $\chi(M) = td_n$ where td_n is an n-th component of the Todd polynomial,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 2 - c_4) + ...$$

REMARK: The Chern classes are obtained as polynomial expression of the curvature (Gauss-Bonnet). Therefore $\chi(\tilde M) = p\chi(M)$ **for any unramified** p**-fold covering** $\tilde M \longrightarrow M$**.**

REMARK: Bochner's vanishing and the classical theory of invariants imply:

1. When $\mathcal{H}\mathrm{ol}(M) = SU(n)$, we have $\dim H^{p,0}(M) = 1$ **for** $p = 1, n$**, and 0 otherwise.** In this case, $\chi(M) = 2$ for even n and 0 for odd.

2. When $\mathcal{H}\mathrm{ol}(M) = Sp(n)$,we have $\dim H^{p,0}(M) = 1$ **for even** p **with** $0 \leqslant p \leqslant 2n$**, and 0 otherwise.** In this case, $\chi(M) = n + 1$.

COROLLARY: $\pi_1(M) = 0$ if $\mathcal{H}\mathrm{ol}(M) = Sp(n)$, or $\mathcal{H}\mathrm{ol}(M) = SU(2n)$. If $\mathcal{H}\mathrm{ol}(M) = SU(2n+1)$, $\pi_1(M)$ **is finite by Cheeger-Gromoll, but can be non-trivial.**