Hyperbolic geometry and the proof of Morrison-Kawamata cone conjecture (2)

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Hyperkähler manifolds

DEFINITION: A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS**, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by the following theorem of Bogomolov.

THEOREM: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

REMARK: Further on, we shall assume (sometimes, implicitly) that **all hyperkähler manifolds we consider are of maximal holonomy**.

The Bogomolov-Beauville-Fujiki (BBF) form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki (BBF) form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

COROLLARY: The space $H^{1,1}(M)$ of I-invariant cohomology classes has signature $(1, b_2 - 2)$ (**hyperbolic signature**).

The Kähler cone and its faces

All results on MBM geometry are **joint work with Ekaterina Amerik**.

DEFINITION: Let M be a compact, Kähler manifold, Kah $\subset H^{1,1}(M,\mathbb{R})$ is Kähler cone (set of all Kähler classes), and $\overline{\text{Kah}}$ its closure in $H^{1,1}(M,\mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\text{Kah}}$ and a hyperplane $V \subset H^{1,1}(M,\mathbb{R})$ which has a non-empry interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)
Let M be a Calabi-Yau manifold. Then the group Aut(M) of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

REMARK: Today I will describe the Kähler cone on holomorphically symplectic manifolds in terms of topological invariants, called **MBM classes**. These are (roughly speaking) classes of minimal rational curves.

Birational contractions and Kawamata bpf

DEFINITION: Base point set of a holomorphic line bundle is an intersection of all zero divisors of its sections. A line bundle with trivial base point set is called **base point free** (bpf). A line bundle L with nL bpf is called **semiample**

CLAIM: Let L be a semiample line bundle on a compact complex variety M. **Then M is equipped with a holomorphic map $\varphi: M \longrightarrow X$ such that $L = \varphi^* L_0$, where L_0 is an ample bundle on X.**

DEFINITION: A line bundle L is **nef** if $c_1(L)$ lies in the closure of the Kähler cone, and **big** if $\int_M c_1(L)^{\dim_{\mathbb{C}} M} > 0$.

THEOREM: (Kawamata bpf theorem; very weak form)
Let L be a nef line bundle on M such that $nL - K_M$ is big. **Then L is semiample.**

For Calabi-Yau manifolds this means just that **big and nef bundles are semiample.**

Birational contractions

DEFINITION: Birational contraction of a complex manifold is a holomorphic birational map $M \longrightarrow X$ to a complex variety X.

REMARK: From Kawamata bpf it follows that **any big and nef bundle L on Calabi-Yau is obtained as $L = \varphi^* L_0$, where $\varphi: M \longrightarrow X$ is a birational contraction and L_0 an ample bundle on X.**

REMARK: Let M be a hyperkähler manifold, η the cohomology class of an extremal curve, ω_0 an integer point on the corresponding face of the Kähler cone, and L the holomorphic line bundle with $c_q(L) = \omega_0$. Then L is big and nef. Then the corresponding birational contraction **contracts all curves C with $[C] = \lambda\eta$.** Indeed, $\langle L_1, C\rangle = 0$.

The Teichmüller space

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the (infinite-dimensional) space of all complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: When M is Calabi-Yau, Teich is **a finite-dimensional complex space**, but often **non-Hausdorff**.

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

DEFINITION: Let Γ^I be the subgroup of Γ preserving a connected component Teich^I of Teich. Then Γ^I is called **monodromy group of (M, I)**.

REMARK: Monodromy group can be obtained as a group generated by monodromy for all Gauss-Manin systems on families of deformations of (M, I). **It has finite index in Γ.**

Birational geometry of hyperkähler manifolds

REMARK: Let (M,I) be a hyperkähler manifold, and $\varphi: (M,I) \dashrightarrow (M,I')$ a bimeromorphic map to another hyperkähler manifold. Since the canonical bundle of (M,I) and (M,I') is trivial, φ is an isomorphism in codimension 1. **This allows one to identify $H^2(M,I)$ and $H^2(M,I')$.** Further on, we call (M,I') "a birational model" for (M,I), and identify $H^2(M)$ for all birational (and bimeromorphic) models.

THEOREM: (Huybrechts)
Bimeromorphic hyperkähler manifolds are diffeomorphic. Moreover, they occur as **non-separate (non-Hausdorff) points on the corresponding Teichmüller space.** Conversely, **all non-separate points on the Teichmüller space correspond to manifolds which are bimeromorphic.**

MBM classes

THEOREM: (Huybrechts, Boucksom)
Let (M,I) be a hyperkähler manifold, and $\eta \in H^{1,1}(M)$ a nef cohomology class such that $q(\eta,\eta) > 0$. **Then $\langle \eta, C\rangle = 0$ for some rational curve C on (M,I).**

REMARK: "MBM classes" are classes of rational curves which occur this way as obstructions to Kählerness.

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

DEFINITION: Let (M,I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M,I)$ is called **minimal** if for any $\mathbb{Q}$-effective homology classes $z_1, z_2 \in H_{1,1}(M,I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. A negative rational homology class $z \in H_{1,1}(M,I)$ is called **monodromy birationally minimal** (MBM) if $\gamma(z)$ is minimal and $\mathbb{Q}$-effective for one of birational models (M,I') of (M,I), where $\gamma \in O(H^2(M))$ is an element of the monodromy group of (M,I).

MBM classes are deformationally invariant

This property is **deformationally invariant**.

This is the main result of this talk.

THEOREM: Let $z \in H^2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type (1,1) with respect to I and I'. Then η **is MBM in** $(M,I) \Leftrightarrow$ **it is MBM in** (M,I').

DEFINITION: Let P be the set of all real vectors in $H^{1,1}(M,I)$ satisfying $q(v,v) > 0$, where q is the Bogomolov-Beauville-Fujiki form on $H^2(M)$. The **positive cone** $\operatorname{Pos}(M,I)$ as a connected component of P containing a Kähler form. Then $\mathbb{P}\operatorname{Pos}(M,I)$ is a hyperbolic space, and $\operatorname{Aut}(M,I)$ acts on $\mathbb{P}\operatorname{Pos}(M,I)$ by hyperbolic isometries.

REMARK: Let $C \in H^{1,1}(M)$ be a negative class. **Then its orthogonal complement $C^\perp$ bisects the positive cone onto two components.**

MBM classes for Pic$(M) = \mathbb{Z}$

The MBM classes are better understood if the Picard group has rank one and generated by a negative vector (in this case M is non-algebraic).

DEFINITION: A homology class z is called **$\mathbb{Q}$-effective** if Nz is effective (represented by a curve) for some $N \in \mathbb{Z}^{>0}$.

THEOREM: Let (M, I) be a hyperkähler manifold, rk Pic$(M, I) = 1$, and $z \in H_{1,1}(M, I)$ a non-zero negative class. **Then z is monodromy birationally minimal if and only if $\pm z$ is $\mathbb{Q}$-effective.**

Proof. Step 1: Clearly, any negative rational curve in (M, I) represents an MBM class.

Step 2: If (M, I) has no rational curves, **its Kähler cone is equal to the positive cone** (Huybrechts, Boucksom). Therefore, z is orthogonal to a Kähler class, and hence non-effective. ■

REMARK: This argument proves that **MBM classes correspond to faces of a Kähler cone** for rk Pic$(M, I) = 1$.

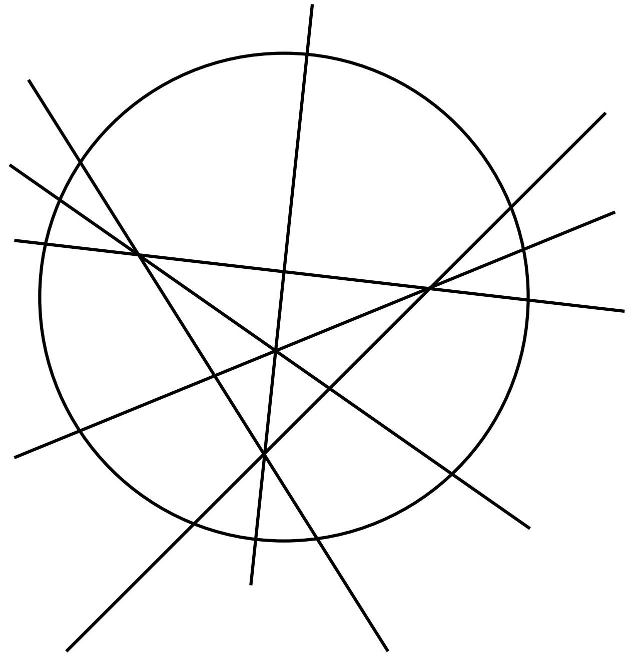
MBM classes and the Kähler cone

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \mid \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

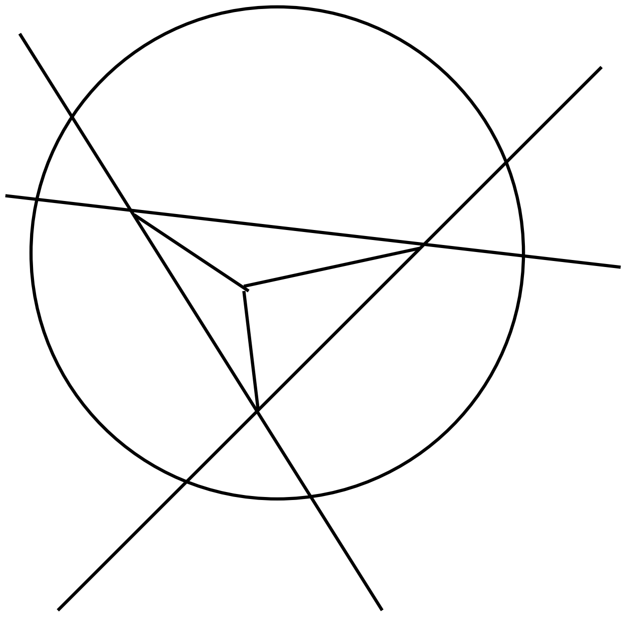
REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

MBM classes and the Kähler cone: the picture

REMARK: This implies that $z^\perp \cap \mathsf{Pos}(M,I)$ **either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM);** that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition



Prohibited partition

Families of rational curves: lower bound on dimension

THEOREM: (Z. Ran)
Let M be a hyperkähler manifold of dimension $2n$. **Then any rational curve $C \subset M$ deforms in a family of dimension at least $2n-2$.**

Proof: By adjunction formula, $\deg(NC) = -2$ and $\operatorname{rk}(NC) = 2n-1$, which implies that C deforms in a family of dimension at least $2n-3$. The extra parameter is due to the existence of the twistor space $\operatorname{Tw}(M)$. This is a complex manifold of dimension $n+1$, fibered over $\mathbb{C}P^1$ in such a way that M is one of the fibers and the other fibers correspond to the other complex structures coming from the hyperkähler action on M. The same adjunction argument shows that C deforms in $\operatorname{Tw}(M)$ in a family of dimension at least $2n-2$. But **all deformations of C are contained in M since the neighbouring fibers contain no curves.** ■

Families of rational curves: coisotropicity

DEFINITION: A complex analytic subvariety Z of a holomorphically symplectic manifold (M, Ω) is called **isotropic** if $\Omega|_Z = 0$ and **coisotropic** if Ω has rank $\frac{1}{2}\dim_{\mathbb{C}} M - \operatorname{codim}_{\mathbb{C}} Z$ on TZ in all smooth points of Z, which is the minimal possible rank for a $2n-p$-dimensional subspace in a $2n$-dimensional symplectic space.

THEOREM: Let M be a hyperkähler manifold, $C \subset M$ a minimal rational curve, and $Z \subset M$ the union of all deformations of C in M. **Then Z is a coisotropic subvariety of M.**

Proof. Step 1: Let V be a MRC quotient of Z. Since fibers of $\pi: Z \longrightarrow V$ are rationally connected, they are isotropic.

Step 2: Let $k := \operatorname{codim} Z$. Let T be the irreducible component of the parameter space for deformations of C in M. We have $dim(T) \geqslant 2n-2$ by Ziv Ran. Therefore the dimension of the universal family of curves over T is at least $2n-1$. Since it projects onto Z which is $2n-k$-dimensional, the fibers of this projection are of dimension at least $k-1$.

Families of rational curves: coisotropicity (2)

THEOREM: Let M be a hyperkähler manifold, $C \subset M$ a minimal rational curve, and $Z \subset M$ the union of all deformations of C in M. **Then Z is a coisotropic subvariety of M.**

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Step 3: By bend-and-break, **there is only a finite number of minimal rational curves through two general points.** This means that the fibers of the MRC fibration $\pi : Z \longrightarrow V$ are at least k-dimensional, and $\dim V \leqslant \dim M - 2k$.

Step 4: Since the fibers of π are isotropic, one has $\operatorname{rk} \Omega|_Z \leqslant \dim V = 1/2 \dim M - k$, hence Z is coisotropic, and the inequality $\dim V \leqslant \dim M - 2k$ is equality. ■

Families of rational curves: upper bound

COROLLARY: The deformation space of minimal rational curves on a holomorphic symplectic manifold is $2n-2$-dimensional.

Proof. Step 1: Let C be a minimal rational curve, and Z the union of all its deformations. Let $k := \operatorname{codim} Z$. Consider the MRC map $\pi: Z \longrightarrow V$. We have shown that $\dim V = \dim M - 2k$.

Step 2: Since $\dim V = \dim M - 2k$, the fibers of $\pi: Z \longrightarrow V$ are k-dimensional. Applying bend-and-break again, we obtain that there is a $2k-2$-dimensional family of deformations of C in each fiber of π. ■

Families of rational curves: deformational invariance (local)

COROLLARY: Let C be a minimal rational curve in a hyperkähler manifold M_0. **Then any small deformation M_t of $M = M_0$ such that the homology class z of C stays of type $(1,1)$ on M_t, contains a deformation of C.**

Proof: From Riemann-Roch theorem it follows that C deforms in a family of dimension at least $2n-3+\dim(\mathrm{Def}(M))$. Since the deformations of C inside any M_t form a family of dimension $2n-2$, the conclusion follows. ■

Families of rational curves: deformational invariance (global)

COROLLARY: If C is minimal, **any deformation M_t of $M = M_0$ such that the corresponding homology class remains of type $(1,1)$, has a birational model containing a rational curve in that homology class.**

Proof: Let $\operatorname{Teich}(M)^0$ be the connected component of the Teichmüller space of M containing the parameter point for our complex manifold M_0, and $\operatorname{Teich}_z(M)^0$ the part of it where z remains of type $(1,1)$. Connecting M_t with M_0 by a path and applying the above corollary, we obtain the proof.

Birational models appear since $\operatorname{Teich}_z(M)$ is not Hausdorff, so that at the end of a path we might arrive to another point of $\operatorname{Teich}_z(M)$, not separable from M_t. However, a theorem of Huybrechts implies that unseparable points of $\operatorname{Teich}_z(M)$ correspond to bimeromorphic complex manifolds. ■

REMARK: **This proves the deformational invariance of MBM classes.**