Hyperbolic geometry and the proof of Morrison-Kawamata cone conjecture (3)

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The Kähler cone and its faces

The work presented here **is done in collaboration with Ekaterina Amerik.**

DEFINITION: Let M be a compact, Kähler manifold, Kah $\subset H^{1,1}(M,\mathbb{R})$ is Kähler cone, and $\overline{\text{Kah}}$ its closure in $H^{1,1}(M,\mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\text{Kah}}$ and a hyperplane $V \subset H^{1,1}(M,\mathbb{R})$ which has a non-empry interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)
Let M be a Calabi-Yau manifold. Then the group Aut(M) of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

(-2)-classes on a K3 surface

CLAIM: (Hodge index theorem)
Let M be a Kähler surface. **Then the form $\eta \longrightarrow \int_M \eta\wedge\eta$ has signature $(+,-,-,...)$ on $H^{1,1}(M,\mathbb{R})$.**

DEFINITION: Positive cone $\operatorname{Pos}(M)$ on a Kähler surface is the one of the two components of

$$\{v\in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \eta\wedge\eta>0\}$$

which contains a Kähler form.

DEFINITION: A cohomology class $\eta\in H^2(M,\mathbb{Z})$ on a K3 surface is called **(-2)-class** if $\int_M \eta\wedge\eta=-2$.

REMARK: Let M be a K3 surface, and $\eta\in H^{1,1}(M,\mathbb{Z})$ a (-2)-class. **Then either η or $-\eta$ is effective.** Indeed, $\chi(\eta)=2+\frac{\eta^2}{2}=1$ by Riemann-Roch.

Kähler cone for a K3 surface

THEOREM: Let M be a K3 surface, and S the set of all effective (-2)-classes. **Then** $\operatorname{Kah}(M)$ **is the set of all** $v \in \operatorname{Pos}(M)$ **such that** $\langle v, s\rangle > 0$ **for all** $s \in S$**.**

Proof: This is a version of Nakai-Moishezon theorem which follows immediately from Demailly-Paun characterization of Kähler classes. ■

DEFINITION: A Weyl chamber on a K3 surface is a connected component of $\operatorname{Pos}(M)\backslash S^\perp$, where $S^\perp$ is a union of all planes $s^\perp$ for all (-2)-classes $s \in S$. **The reflection group** of a K3 surface is a group W generated by reflections with respect to all $s \in S$.

REMARK: Clearly, a Weyl chamber is a fundamental domain of W, and W acts transitively on the set of all Weyl chambers. Moreover, **the Kähler cone of** M **is one of its Weyl chambers.**

Cone conjecture for a K3 surface

THEOREM: Let M be a K3 surface. **Then** $\operatorname{Aut}(M)$ **is the group of all isometries of** $H^{1,1}(M,\mathbb{Z})$ **preserving the Kähler chamber.**

Proof: This result **directly follows from the global Torelli theorem.** ■

COROLLARY: (H. Sterk) Morrison-Kawamata cone conjecture holds for a K3 surface.

Proof. Step 1: A group Γ of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \mid l^2 = x\}$ for any given x (see Kneser, *Quadratische Formen*, Satz 30.2). **Therefore, Γ acts with finitely many orbits on the set of (-2)-vectors in Λ.** This can be used to show that Γ **acts with finitely many orbits on faces of all Weyl chambers.**

Step 2: For each pair of faces F, F' of a Kähler cone and $w \in O(\Lambda)$ mapping F to F', w maps Kah to itself or to an adjoint Weyl chamber K'. Then $K' = r(K)$, where r is the reflection fixing F'. In the first case, $w \in \operatorname{Aut}(M)$. In the second case, rw maps F to F' and maps Kah to itself, hence $rw \in \operatorname{Aut}(M)$.
■

Hyperkähler manifolds (reminder)

DEFINITION: A **hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

DEFINITION: A hyperkähler manifold M is called **of maximal holonomy**, or **IHS**, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

This definition is motivated by the following theorem of Bogomolov.

THEOREM: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

REMARK: Further on, we shall assume (sometimes, implicitly) that **all hyperkähler manifolds we consider are of maximal holonomy**.

The Bogomolov-Beauville-Fujiki form (reminder)

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki (BBF) form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

COROLLARY: The space $H^{1,1}(M)$ of I-invariant cohomology classes has signature $(1, b_2 - 2)$ (**hyperbolic signature**).

Kleinian groups

DEFINITION: **Kleinian group** is a discrete subgroup $\Gamma \subset SO(1,n)$ of finite Haar covolume (that is, **the quotient $SO(1,n)/\Gamma$ has finite volume**).

DEFINITION: An **arithmetic subgroup** of an algebraic group G is a finite index subgroup in $G_{\mathbb{Z}}$.

REMARK: From Borel and Harish-Chandra, it follows that **any arithmetic subgroup of $SO(1,n)$ is Kleinian, for $n \geqslant 2$.**

DEFINITION: Let V be a real space equipped with a quadratic form of signature $(1,n)$. A **hyperbolic orbifold** is a quotient of $\mathbb{P}^+(V)$ (projectivisation of a positive cone) by a Kleinian subgroup of $SO(V)$.

REMARK: The space $\mathbb{P}^+(V)$ is equipped with a unique (up to a scalar factor) $SO(1,n)$-invariant Riemannian metric. We consider a hyperbolic orbifold as a Riemannian orbifold, equipped with this metric, which is called **the hyperbolic metric**.

Monodromy group

From Eyal Markman's "Survey of Torelli theorem...": some consequences of global Torelli.

DEFINITION: **Monodromy group** $\operatorname{Mon}(M)$ of a hyperkähler manifold (M, I) is a subgroup of $O(H^2(M,\mathbb{Z}), q)$ generated by monodromy of Gauss-Manin connections for all families of deformations of (M, I). The **Hodge monodromy group** $\operatorname{Mon}(M, I)$ is a subgroup of $\operatorname{Mon}(M)$ preserving the Hodge decomposition.

THEOREM: $\operatorname{Mon}(M)$ **is an arithmetic subgroup of** $SO(H^2(M,\mathbb{R}), q)$**.**

DEFINITION: Let (M, I') be a holomorphic symplectic manifold pseudo-isomorphic to (M, I). A **Kähler chamber** of (M, I) is an image of the Kähler cone of (M, I') under the action of $\operatorname{Mon}(M, I)$.

CLAIM: $\operatorname{Mon}(M, I)$ acts on $H^{1,1}(M, I)$ **mapping Kähler chambers to Kähler chambers.**

CLAIM: The group of automorphisms $\operatorname{Aut}(M, I)$ **is a group of all elements of** $\operatorname{Mon}(M, I)$ **preserving the Kähler cone.**

Positive cone

DEFINITION: Let P be the set of all real vectors in $H^{1,1}(M,I)$ satisfying $q(v,v) > 0$, where q is the Bogomolov-Beauville-Fujiki form on $H^2(M)$. The **positive cone** $\operatorname{Pos}(M,I)$ as a connected component of P containing a Kähler form. Then $\mathbb{P}\operatorname{Pos}(M,I)$ is a hyperbolic space, and $\operatorname{Mon}(M,I)$ acts on $\mathbb{P}\operatorname{Pos}(M,I)$ by hyperbolic isometries.

THEOREM: **The positive cone is partitoned onto Kähler chambers.** Interiors of different Kähler chambers are disjoint, the closure of their union contains the positive cone.

DEFINITION: Let $H^{1,1}(M,\mathbb{Q})$ be the set of all rational (1,1)-classes on (M,I), and $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ the set of all Kähler classes in $H^{1,1}(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. Then $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ is called **ample cone** of M.

Hyperbolic manifolds associated with a hyperkähler manifold

REMARK: From global Torelli theorem it follows that $\operatorname{Mon}(M,I)$ is a finite index subgroup in $O(H^2(M,\mathbb{Z}),q)$. Therefore, $\operatorname{Mon}(M,I)$ **acts on** $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I) := \mathbb{P}(\operatorname{Pos}(M,I) \cap H^{1,1}(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R})$ **with finite covolume;** in other words, $\operatorname{Mon}(M,I)$ is Kleinian, and the quotient $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$ is a finite volume hyperbolic orbifold.

REMARK: Notice that $\operatorname{Aut}(M,I)$ is a stabilizer of $\operatorname{Kah}(M)$ in $\operatorname{Mon}(M,I)$.

THEOREM: **(cone conjecture)**
The quotient $\operatorname{Kah}_{\mathbb{Q}}(M,I)/\operatorname{Aut}(M,I)$ is a finite hyperbolic polyhedron in $\mathbb{P}\operatorname{Pos}_{\mathbb{Q}}(M,I)/\operatorname{Mon}(M,I)$.

REMARK: In other words, **the action of $\operatorname{Aut}(M,I)$ on $\operatorname{Kah}_{\mathbb{Q}}(M,I)$ has a finite polyhedral fundamental domain.**

MBM classes (reminder)

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

DEFINITION: Let (M,I) be a (non-algebraic) hyperkähler manifold with the Pocard group $H^{1,1}(M,\mathbb{Z})$ generated by a negative class $\eta \in H^2(M,\mathbb{Z})$. The class η is called **MBM** if (M,I) contains a curve C.

The MBM property is in fact deformational invariant:

THEOREM: Let $z \in H^2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type (1,1) with respect to I and I'. Then η **is MBM in** $(M,I) \Leftrightarrow$ **it is MBM in** (M,I').

DEFINITION: Let $z \in H^2(M,\mathbb{Z})$ be a negative class on a hyperkähler manifold (M,I). It is called **an MBM class** if for any complex structure I' in the same deformation class satisfying $z \in H^{1,1}(M,I')$, z is an MBM class.

MBM classes and the Kähler cone (reminder)

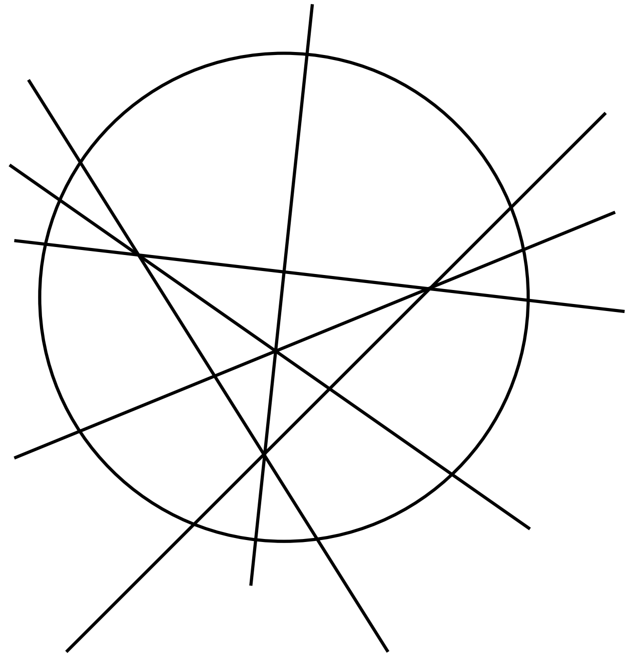
THEOREM: (Amerik-V.) Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \ | \ \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

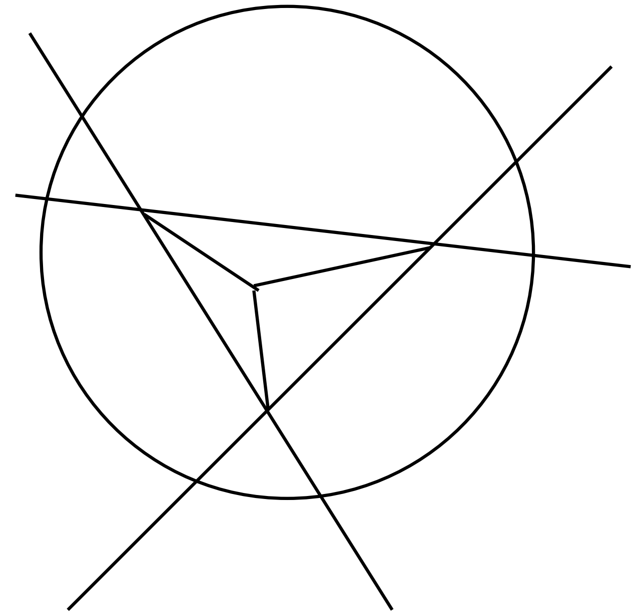
THEOREM: (Morrison-Kawamata cone conjecture)
The group $\operatorname{Mon}(M,I)$ acts on the set of faces of the Kähler cone with finitely many orbits.

REMARK: This would follow if we prove that $\operatorname{Mon}(M,I)$ acts on MBM classes with finitely many orbits.

MBM classes and the Kähler cone: the picture



Allowed partition



Prohibited partition

MBM classes and the cone conjecture

Theorem 1: Let (M,I) be a hyperkähler manifold, and $\{s_i\}$ the set of MBM classes of type (1,1). **Then** Mon(M,I) **acts on** $\{s_i\}$ **with finitely many orbits.**

COROLLARY: (Morrison-Kawamata cone conjecture)
The group Aut(M,I) **acts on the ample cone with finte polyhedral fundamental domain.**

Proof: The quotient Kah$(M,I)/$ Aut(M,I) is a finite polyhedron in Pos$(M,I)/$ Mon(M,I). ∎

REMARK: Theorem 1 is immediately implied by the following result of hyperbolic geometry.

Theorem 2: Let X be a hyperbolic manifold of dimension > 2, and $\{S_i\}$ an infinite set of geodesic hypersurfaces. Then **either this set is finite, or** $\bigcup S_i$ **is dense in** X**.**

Ratner's orbit closure theorem

DEFINITION: Let G be a Lie group, and $\Gamma \subset G$ a discrete subgroup. We say that Γ **has finite covolume** if the Haar measure of G/Γ is finite. In this case Γ is called **a lattice subgroup**.

REMARK: Borel and Harish-Chandra proved that an arithmetic subgroup of a reductive group G is a lattice whenever G has no non-trivial characters over $\mathbb{Q}$. In particular, **all arithmetic subgroups of a semi-simple group are lattices.**

DEFINITION: Let G be a Lie group, and $g \in G$ any element. We say that g is **unipotent** if $g = e^h$ for a nilpotent element h in its Lie algebra. A group G is **generated by unipotents** if G is multiplicatively generated by unipotent one-parameter subgroups.

THEOREM: (Ratner orbit closure theorem)
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. **Then the closure of any H-orbit Hx in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap x\Gamma x^{-1} \subset S$ is a lattice in S.**

Ratner's measure classification theorem

DEFINITION: Let (M,μ) be a space with a measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M')=0$ or $\mu(M\backslash M')=0$.

REMARK: Ergodic measures are extremal rays in the cone of all G-invariant measures.

REMARK: By Choquet's theorem, **any G-invariant measure on M is expressed as an average of a certain set of ergodic measures.**

DEFINITION: Let G be a Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider an orbit $S\cdot x \subset G$ of a closed subgroup $S\subset G$, put the Haar measure on $S\cdot x$, and assume that its image in G/Γ is closed. A measure on G/Γ is called **algebraic** if it is proportional to the pushforward of the Haar measure on $S\cdot x/\Gamma$ to G/Γ.

THEOREM: (Ratner's measure classification theorem)
Let G be a connected Lie group, Γ a lattice, and G/Γ the quotient space, considered as a space with Haar measure. Consider a finite measure μ on G/Γ. Assume that μ is invariant and ergodic with respect to an action of a subgroup $H\subset G$ generated by unipotents. **Then μ is algebraic.**

Mozes-Shah and Dani-Margulis

THEOREM: (Mozes-Shah)
A limit μ of a sequence μ_i of algebraic measures is again an algebraic measure. Moreover, **if the support of μ has the same dimension as μ_i, this sequence stabilizes.**

Proof: Follows from Ratner's measure classification theorem. ■

DEFINITION: A measure μ on M is called **probabilistic** if $\mu(M) = 1$.

THEOREM: (Dani-Margulis)
Let μ_i be a converging sequence of probabilistic algebraic measures on a Lie group G, associated with subgroups $S_i \subset G$ generated by unipotents, and $C \subset G$ a compact subset such that $\mu_i(C) > \varepsilon$ for some $\varepsilon > 0$. **Then μ_i converges to a probabilistic measure on G.**

REMARK: The space of measures with $\mu(M) \leqslant 1$ is compact, but **the limit of probabilistic measures is not generally probabilistic.**

Geodesic hypersurfaces in hyperbolic manifolds

THEOREM: Let X be a complete Riemannian orbifold of dimension at least 3, constant negative curvature and finite volume, and $\{S_i\}$ a set of infinitely many complete, locally geodesic hypersurfaces. **Then the union of S_i is dense in X.**

Proof. Step 1: The group $SO(1, n-1)$is generated by unipotents. Therefore, Ratner's theorem can be applied to S_i which are orbits of $SO(1, n-1)$. Any subgroup of $SO(1, n)$ strictly containing $SO(1, n-1)$ coincides with $SO(1, n)$. By Ratner's theorem, either S_i is closed and has finite volume, or it is dense. Therefore, **we may assume that S_i is a closed hyperbolic hypersurface in X.**

Step 2: Denote by μ_i the probabilistic algebraic measure supported in S_i. Using the structure theorem for cusps, we obtain that the support of all μ_i intersects a certain compact $K \subset X$. Using Dani-Margulis theorem, we obtain that μ_i has a subsequence converging to an algebraic measure μ. By Moses-Shah, **μ is supported in an orbit of a subgroup H_1 strictly containing $SO(1, n-1)$.**

Step 3: By Step 1, $H_1 = SO(1, n)$ ■