Hyperbolic geometry and the proof of Morrison-Kawamata cone conjecture (4)

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Holomorphically symplectic manifolds (reminder)

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form (reminder)

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c>0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda>0$.

Remark: **q has signature $(3, b_2-3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Automorphisms of cohomology.

THEOREM: Let M be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then G acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G \longrightarrow O(H^2(M,\mathbb{R}),q)$ **is surjective on a connected component, and has compact kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = cq(v,v)^n$ implies that G **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, G preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any non-trivial stable bundle with $c_1(B) = 0$.

Step 3: $\mathfrak{o}(H^2(M,\mathbb{R}),q)$ acts on $H^*(M,\mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore Lie(G) surjects to $\mathfrak{o}(H^2(M,\mathbb{R}),q)$.

Step 4: The kernel K of the map $G \longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form. ■

Sullivan's theorem

Theorem: (Dennis Sullivan)
Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: Follows from the computation of $G = \operatorname{Aut}(H^*(M,\mathbb{R}),p_1,...,p_n)$ done earlier. Indeed, the kernel of $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ is a set of integer points of a compact Lie group, hence finite. The image of $\Gamma_0 = G_{\mathbb{Z}}$ has finite index in $O(H^2(M,\mathbb{Z}),q)$, because the corresponding map of Lie groups is surjective. ■

Computation of the mapping class group

COROLLARY: The mapping class group **Γ is mapped to $O(H^2(M,\mathbb{Z}),q)$ with finite kernel and finite index.**

Proof: By Sullivan, Γ is mapped to Γ_0 with finite kernel and finite index, and $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ has finite kernel and finite index, as shown above. ■

THEOREM: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I has finite index in Γ.**

REMARK: Γ_I is a group generated by monodromy of all Gauss-Manin local systems for all deformations of (M,I). It is known as **the monodromy group** of (M,I).

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\text{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\text{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: (Bogomolov)
Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then **The period map P : Teich $\longrightarrow \mathbb{P}er$ is etale.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **usually non-Hausdorff**.

Birational equivalence and non-separable points

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separable points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2).

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: (**Global Torelli theorem**)
Let (M, I) be a hyperkähler manifold, and Teich_b^I a connected component of its birational Teichmüller space. **Then Teich_b^I is isomorphic to $\mathbb{P}\mathrm{er}$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$.** Two points in Teich_b^I coorespond to birational manifolds if they lie in the same Γ^I-orbit, where Γ^I is the monodromy group. Finally, **Γ^I is an arithmetic lattice in $SO(b_2 - 3, 3)$.**

The group of symplectic Hodge monodromy

DEFINITION: Let (M,I) be a hyperkähler manifold. Then **the Hodge monodromy group** $\operatorname{Mon}_I(M)$ is the group of all $a\in\operatorname{Mon}(M)$ preserving the Hodge decomposition on $H^2(M)$.

DEFINITION: Let Ω be a holomorphic symplectic form on a hyperkähler manifold. Consider the homomorphism $\varphi:\ \operatorname{Mon}_I(M)\longrightarrow\mathbb{C}^*$, $\varphi(\gamma)=\frac{\gamma^*\Omega}{\Omega}$. Denote its kernel by $\operatorname{Mon}_{I,\Omega}(M,I)$. Thi group is called **the group of symplectic Hodge monodromy.**

Claim 1: Consider the Hodge lattice $\Lambda:=H^{1,1}_I(M,\mathbb{Z})$. **Then the natural homomorphism $\operatorname{Mon}_{I,\Omega}(M,I)\longrightarrow O(\Lambda)$ is injective and has finite index.**

Proof: Let $H^2_{tr}(M):=H^{1,1}_I(M,\mathbb{Q})^\perp$ be the "transcendental part" of the Hodge lattice, that is, the smallest Hodge substructure containing $\operatorname{Re}H^{2,0}(M)$. By definition,

$$\operatorname{Mon}_{I,\Omega}(M,I)=\left\{a\in\operatorname{Mon}(M)\ \middle|\ a\Big|_{H^2_{tr}(M)}=\operatorname{Id}\right\}$$

Since $\operatorname{Mon}(M)$ is an arithmetic lattice subgroup in $O(H^2(M,\mathbb{Z}))$, $\operatorname{Mon}_{I,\Omega}(M,I)$ is an arthmetic lattice in the group of isometries of $H^2_{tr}(M)^\perp=\Lambda$. ■

MBM classes (reminder)

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H_2(M, \mathbb{R}) = H^2(M, \mathbb{R})$ satisfying $q(\eta, \eta) < 0$. It is **effective** if it is represented by a curve.

THEOREM: Let $z \in H_2(M, \mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that z is of type (1,1) with respect to I and I' and $\operatorname{Pic}(M) = \langle z \rangle$. Then **$\pm z$ is effective in (M, I) $\Leftrightarrow$ iff it is effective in (M, I').**

REMARK: From now on, we identify $H^2(M)$ and $H_2(M)$ using the BBF form. Under this identification, **integer classes in $H_2(M)$ correspond to rational classes in $H^2(M)$** (the form q is not unimodular).

DEFINITION: A negative class $z \in H^2(M, \mathbb{Z})$ on a hyperkähler manifold is called **an MBM class** if there exist a deformation of M with $\operatorname{Pic}(M) = \langle z \rangle$ such that λz is represented by a curve, for some $\lambda \neq 0$.

MBM classes and the shape of the Kähler cone (reminder)

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \mid z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

DEFINITION: **Kähler chamber** is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$.

CLAIM: The Hodge monodromy group maps Kähler chambers to Kähler chambers.

MBM classes and automorphisms

THEOREM: Let (M, I) be a hyperkähler manifold, $\operatorname{Mon}(M)$ the group of automorphisms of $H^2(M)$ generated by monodromy transform for all Gauss-Manin local systems, and $\operatorname{Mon}_I(M)$ the Hodge monodromy group, that is, a subgroup of $\operatorname{Mon}(M)$ preserving the Hodge decomposition. **Then** $\operatorname{Aut}(M)$ **is a subgroup of** $\operatorname{Mon}_I(M)$ **preserving the Kähler cone** $\operatorname{Kah}(M)$**.**

COROLLARY: Let (M, I) be a hyperkähler manifold such that there are no MBM classes of type (1,1). **Then** $\operatorname{Aut}(M) = \operatorname{Mon}_I(M)$**.**

Proof: Indeed, for such manifold $\operatorname{Kah}(M, I) = \operatorname{Pos}(M, I)$. ■

Morrison-Kawamata cone conjecture

DEFINITION: An integer cohomology class a is **primitive** if it is not divisible by integer numbers $c > 1$.

THEOREM: (a version of Morrison-Kawamata cone conjecture) **The group $\operatorname{Mon}(M)$ acts on the set of primitive MBM classes with finitely many orbits.**

Proof: Proven by Amerik-V., using homogeneous dynamics (Ratner theorems, Dani-Margulis, Mozes-Shah). ■

COROLLARY: Let M be a hyperkähler manifold. Then there exists a number $N > 0$, called **an MBM bound**, such that any MBM class z satisfies $|q(z,z)| < N$.

Proof: There are only finitely many primitive MBM classes, up to isometry action, and the have finitely many squares. ■

Corollary 1: Let M be a hyperkähler manifold, N its MBM bound, and (M,I) a deformation such that for any $x \in H^{1,1}_I(M,\mathbb{Z})$ one has $q(x,x) > N$. **Then (M,I) has no MBM classes of type (1,1), and $\operatorname{Kah}(M,I) = \operatorname{Pos}(M,I)$ and $\operatorname{Aut}(M) = \operatorname{Mon}_I(M)$.** ■

Classification of automorphisms of a hyperbolic space

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \mid q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$. (α is **"parabolic isometry"**)

DEFINITION: An automorphism of a hyperkähler manifold (M,I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M,\mathbb{R})$.

Primitive sublattices with an MBM bound

DEFINITION: Integer lattice, or **quadratic lattice**, or just **lattice** is $\mathbb{Z}^n$ equipped with an integer-valued quadratic form. **When we speak of embedding of lattices, we always assume that they are compatible with the quadratic form.**

DEFINITION: A sublattice $\Lambda' \subset \Lambda$ is called **primitive** if $(\Lambda' \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \Lambda = \Lambda'$. A number a is **represented** by a lattice (Λ, q) if $a = q(x,x)$ for some $x \in \Lambda$. **Minumum** of a lattice is the number $\min \Lambda := \min_x |q(x,x)|$, taken over all $x \in \Lambda$.

Theorem 1: Let (Λ, q) be a lattice of signature (n,m), $n \geqslant 3, m \geqslant 2$. Fix a number $N > 0$. **Then there exists a primitive sublattice $\Lambda' \subset \Lambda$ of rank 2, signature $(1,1)$ with $\min \Lambda' > N$.**

Proof: Takes some number theory (Hilbert symbols, quadratic residues). For unimodular lattices it is Witt-Nikulin theorem.

DEFINITION: Let M be a hyperkähler manifold, $\Lambda = H^2(M,\mathbb{Z})$, q the BBF form. A primitive sublattice $\Lambda' \subset H^2(M,\mathbb{Z})$ **satisfies MBM bound** if its minimum is $> N$, where N is the MBM bound of M.

Sublattices with MBM bound and automorphisms

REMARK: By Torelli theorem, **for any primitive sublattice $\Lambda \subset H^2(M, \mathbb{Z})$, there exists a complex structure I such that $\Lambda = H^{1,1}_I(M, \mathbb{Z})$,** if $H^2(M, \mathbb{R})/(\Lambda \otimes_{\mathbb{Z}} \mathbb{R})$ has signature (p, q) with $p \geqslant 2$.

THEOREM: Let M be a hyperkähler manifold, and $\Lambda \subset H^2(M, \mathbb{Z})$ a primitive sublattice satisfying the MBM bound. Let (M, I) be a deformation of M such that $\Lambda = H^{1,1}_I(M, \mathbb{Z})$. **Then the group of holomorphic symplectic automorphisms $\operatorname{Aut}(M, \Omega) = \operatorname{Mon}_{I,\Omega}(M)$ surjects to a subgroup of finite index in $O(\Lambda)$.**

Proof: Since $\Lambda = H^{1,1}_I(M, \mathbb{Z})$ satisfies MBM bound, it contains no MBM classes. **By Corollary 1, this gives $\operatorname{Aut}(M, \Omega) = \operatorname{Mon}_{I,\Omega}(M)$.** Now, $\operatorname{Mon}_{I,\Omega}(M)$ is a finite index subgroup in $O(\Lambda)$, as follows from Claim 1. ■

Existence of hyperbolic automorphisms

THEOREM: Let M be a hyperkähler manifold, with $b_2(M) \geqslant 7$. Then M has a deformation admitting a hyperbolic automorphism.

Proof. Step 1: Find a primitive rank 2 sublattice $\Lambda \subset H^2(M, \mathbb{Z})$ satisfying the MBM bound. Using Torelli theorem, we construct a deformation M' of M which has $\Lambda = H^{1,1}(M') \cap H^2(M', \mathbb{Z})$.

Proof. Step 2: For such M', the group of symplectic automorphisms surjects to a finite index subgroup of $O(\Lambda)$.

Proof. Step 3: If Λ is rank 2, signature (1,1) quadratic lattice not representing 0, the group $O(\Lambda)$ has infinite order (follows from Dirichlet unit theorem).
■