Kähler manifolds,

lecture 1

Misha Verbitsky

Harish-Chandra Research Institute

January 11, December 2010,

Allahabad.

Complex action on vector spaces

Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. **We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity:** $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues of I are $\pm\sqrt{-1}$.**

2. **V admits an I-invariant metric g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

4. **All eigenvalues of I are equal to $\pm\sqrt{-1}$.** Indeed, $I^2 = -1$.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.** Indeed, I is real.

The Hodge decomposition in linear algebra

DEFINITION: The Hodge decomposition $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q}V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p}V$.

Further on, **TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i-forms.** It is a Grassman algebra on TM

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that $\mathbb{C}P^n$ **is a complex manifold,** in the sense of the above definition.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, by invariants theory.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

Connection and torsion

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

DEFINITION: A torsion of a connection $\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M$ is a map $\operatorname{Alt} \circ \nabla - d$, where $\operatorname{Alt}: \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication. It is a map $T_\nabla: \Lambda^1 M \longrightarrow \Lambda^2 M$.

An exercise: Prove that torsion is a $C^\infty M$-linear.

Linearized torsion map

DEFINITION: A torsor over a group G is a space X with a free, transitive action of G.

EXAMPLE: An affine space is a torsor over a linear space.

REMARK: If ∇_1 and ∇_2 are connections B, the difference $\nabla_{-}\nabla_2$ is $C^\infty M$-linear. **This makes the space $\mathcal{A}(B)$ of connections on B into an affine space,** that is, a torsor over a linear space $\Lambda^1(M) \otimes \operatorname{End}(B)$.

REMARK: Torsion is an affine map

$$\mathcal{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

DEFINITION: An linearized torsion map is a map

$$T_{\nabla, lin}: \ \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow TM \otimes \Lambda^2 M$$

obtained as a linearization of a torsion map $\mathcal{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M)$.

REMARK: It is equal to

$$\operatorname{Alt} \otimes \operatorname{Id}_{TM}: \ \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM.$$

Orthogonal connection

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

CLAIM: Orthogonal connection always exists, on any vector bundle B.

Proof: Take a covering $\{U_i\}$ such that $B\big|_{U_i}$ are trivial and admit an orthonormal frame. Choose connections ∇_i locally on U_i fixing these frames. Then patch the local pieces together, using a splitting ψ_i of unit:

$$\nabla(b) = \sum \nabla_i(\psi_i b)$$

Exercise: Show that this defines a connection.

REMARK: Let ∇, ∇' be two connections. **Their difference is $C^\infty M$-linear:** $\nabla - \nabla' \in \Lambda^1 \otimes \operatorname{End} TM$. If both connections ∇, ∇' are orthogonal, one has $\nabla - \nabla'(g) = 0$. **This means that $\nabla - \nabla' \in \Lambda^1 \otimes \mathfrak{o}(TM)$,** where $\mathfrak{o}(TM)$ denotes the space of antisymmetric endomorphisms.

Levi-Civita connection

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Proof: Choose any orthogonal connection ∇_0. The space of all orthogonal connections is affine space modeled on $\Lambda^1 M \otimes \mathfrak{o}(TM)$.

Step 1: Identifying TM and $\Lambda^1 M$, obtain $\mathfrak{o}(TM) = \Lambda^2 M$.

Step 2: The linearized torsion map is

$$\mathsf{Alt} \otimes \mathsf{Id}_{TM} : \ \Lambda^1(M) \otimes \Lambda^2 M \longrightarrow \Lambda^2 M \otimes TM.$$

This is an isomorphism (dimension count, representation theory). Denote it by Ψ.

Step 3: Take $\nabla := \nabla_0 - \Psi^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - \Psi(\Psi^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free**. ■

Levi-Civita connection on a Kähler manifold

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMARK: The implication (ii) $\Rightarrow$ (i) is clear. Indeed, $[X, Y] = \nabla_X Y - \nabla_Y X$, hence it is a $(1,0)$-vector field when X, Y are of type (1,0), and then **I is integrable**. Also, **$d\omega = 0$, because ∇ is torsion-free,** and $d\omega = \operatorname{Alt}(\nabla\omega)$.

Let us prove (i) $\Rightarrow$ (ii). Step 1: For an almost complex Hermitian structure, choose a connection ∇_0 preserving I and g. **A difference between such connections lies in $\Lambda^1 \otimes \mathfrak{u}(TM)$,** where $\mathfrak{u}(TM)$ is the bundle of skew-Hermitian endomorphisms on TM.

Step 2: We identify $\mathfrak{u}(TM)$ and $\Lambda^{1,1}M$. Then, the linearized torsion map for a Hermitian connection on an almost complex manifold is given by

$$T_{\nabla,lin}: \Lambda^1(M) \otimes \Lambda^{1,1}(M) \longrightarrow \Lambda^2 M \otimes \Lambda^1(M).$$

Linearized torsion of a Hermitian manifold

Step 3: The torsion of ∇_0 belongs to

$$\Lambda^{1,1}(M) \otimes \Lambda^1(M) \oplus \Lambda^{2,0} \otimes \Lambda^{0,1}(M) \oplus \Lambda^{0,2} \otimes \Lambda^{1,0}(M)$$

because ∇_0 **preserves the Hodge decomposition, and I is integrable:**

$$T_{\nabla_0}(X,Y) = \nabla_{0X} Y - \nabla_{0Y} X$$

Step 4: The linearized torsion map induces an exact sequence

$$\Lambda^1(M) \otimes \Lambda^{1,1}(M) \stackrel{\Psi}{\longrightarrow} \Lambda^{1,1} M \otimes \Lambda^1(M) \oplus \Lambda^{2,0} \otimes \Lambda^{0,1}(M) \oplus \Lambda^{0,2} \otimes \Lambda^{1,0}(M) \stackrel{\mathsf{Alt}}{\longrightarrow} \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M),$$

where Alt is the antisymmetrization map (dimension count).

Step 5: $0 = d\omega = T_{\nabla_0}(\omega) - \nabla(\omega) = T_{\nabla_0}(\omega)$. This means that **the antisymmetrization of T_{∇_0} vanishes.**

Step 6: From the above exact sequence it follows that $T_{\nabla_0} \subset \operatorname{im}(\Psi)$.

Step 7: **Then $\nabla := \nabla_0 - \Psi^{-1}(T_{\nabla_0})$ has zero torsion.**