Kähler manifolds

lecture 2

Misha Verbitsky

Harish-Chandra Research Institute

January 13, December 2010,

Allahabad.

Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra**(or "graded associative algebra") is an associative algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded paces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$-action,** with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$-action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then** $\operatorname{End}(A^*), \{\cdot,\cdot\}$ **is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even element. **Then** $\{\{L,d\},d\} = 0$.

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + \{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

de Rham differential

DEFINITION: Let M be a real manifold, and $\Lambda^* M$ the space of differential forms on M. We define **the de Rham differential** as the only operator $d: \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$ which satisfies

1. **The graded Leibniz identity:** $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d\beta$

2. $d^2 = 0$

3. On functions, $d: C^\infty M \longrightarrow \Lambda^1(M)$ is the usual differential.

REMARK: Uniqueness of d is clear. Indeed, d is determined by its values on any set of multiplicative generators of $\Lambda^*(M)$. On the other hand, $\Lambda^*(M)$ is generated by $C^\infty M$ and $dC^\infty M$.

Existence of d: it suffices to prove that d exists locally, then patch together local differentials by uniqueness. On $\mathbb{R}^n$, one has $d(fP) = \sum_i \frac{df}{dx_i} dx_i \wedge P$ for every coordinate monomial form $P = dx_{i_1} \wedge ... \wedge dx_{i_k}$.

DEFINITION: A form η is called **closed** if $d\eta = 0$, and **exact** if $\eta = d\xi$.

Hodge $*$ operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing:** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n-manifold. Define **the Hodge $*$ operator** $* : \Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge * \beta$.

REMARK: The Hodge $*$ operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, ..., \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge ... \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, ..., \xi_{i_k}$, and s the signature of a permutation $(i_1, ..., i_k, j_1, ..., j_{n-k})$.

REMARK: $*^2\big|_{\Lambda^k(M)} = (-1)^{k(n-k)} \operatorname{Id}_{\Lambda^k(M)}$

Hodge theory

CLAIM: On a compact Riemannian n-manifold, one has $d^*\Big|_{\Lambda^k M} = (-1)^{nk} * d *$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d(\beta),$$

one has $(d\alpha, *\beta) = (-1)^{\tilde{\alpha}}(\alpha, *d\beta)$. Setting $\gamma := *\beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde{\alpha}}(\alpha, *d(*)^{-1}\gamma) = (-1)^{\tilde{\alpha}}(-1)^{\tilde{\alpha}(\tilde{n}-\tilde{\alpha})}(\alpha, *d*\gamma) = (-1)^{\tilde{\alpha}\tilde{n}}(\alpha, *d*\gamma).$$

■

DEFINITION: The anticommutator $\Delta := \{d, d^*\} = dd^* + d^*d$ is called **the Laplacian** of M. It is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$.

THEOREM: (The main theorem of Hodge theory) **There is a basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ.**

THEOREM: ("Elliptic regularity for Δ") Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ. **Then α is a smooth k-form.**

De Rham cohomology

DEFINITION: The space $H^i(M) := \frac{\ker d\big|_{\Lambda^i M}}{d(\Lambda^{i-1}M)}$ is called **the de Rham cohomology of M**.

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^* x, d^* x)$, hence $\alpha \in \ker d \cap \ker d^*$

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that **Δ commutes with the de Rham differential.**

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \hat{=} \bigoplus_\alpha \mathcal{H}^*_\alpha(M)$, where α runs through all eigenvalues of Δ, and $\mathcal{H}^*_\alpha(M)$ is the corresponding eigenspace. **For each α, de Rham differential defines a complex**

$$\mathcal{H}^0_\alpha(M) \xrightarrow{d} \mathcal{H}^1_\alpha(M) \xrightarrow{d} \mathcal{H}^2_\alpha(M) \xrightarrow{d} ...$$

Step 3: On $\mathcal{H}^*_\alpha(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that **the complexes $(\mathcal{H}^*_\alpha(M), d)$ don't contribute to cohomology.**

Step 4: We have proven that

$$H^*(\Lambda^*M, d) = \bigoplus_\alpha H^*(\mathcal{H}^*_\alpha(M), d) = H^*(\mathcal{H}^*_0(M), d) = \mathcal{H}^*(M).$$

■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. $\mathcal{I}\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a 9-dimensional Lie superalgebra $\mathfrak{a}$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.