Kähler manifolds

lecture 3

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Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or "supercommutative") if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\} : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity** $\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\operatorname{End}(A^*), \{\cdot,\cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even element. **Then $\{\{L,d\},d\} = 0$.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + \{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. **The Hodge operator** $L(\alpha) := \omega \wedge \alpha$

2. **The Hodge operator** $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. **The Weil operator:** $\mathcal{W}\big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$. **This operator is real.**

THEOREM: These operators generate a 9-dimensional Lie superalgebra $\mathfrak{a}$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

Reference:

JM Figueroa-O'Farrill, C Koehl, B Spence, **Supersymmetry and the cohomology of (hyper)Kaehler manifolds,** arXiv:hep-th/9705161, Nucl.Phys. B503 (1997) 614-626

M. Verbitsky, **Hyperkaehler manifolds with torsion, supersymmetry and Hodge theory,** arXiv:math/0112215, Asian J. Math. Vol. 6, No. 4, pp. 679-712 (2002)

Elena Poletaeva, **Superconformal algebras and Lie superalgebras of the Hodge theory**, arXiv:hep-th/0209168, J.Nonlin.Math.Phys. 10 (2003) 141-147

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and $v_1, ..., v_{2n}$ an orthonormal basis. Denote by $e_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = e_i \wedge \eta$. Let $i_{v_i}: \Lambda^k V \longrightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = * e_{v_i} *$.

CLAIM: The operators e_{v_i}, i_{v_i}, Id are a basis of an **odd Heisenberg Lie superalgebra $\mathfrak{H}$**, with **the only non-trivial supercommutator given by the formula $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \operatorname{Id}$.**

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: From the commutator relations in $\mathfrak{H}$, one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}}\right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k - n$ on k-forms.

The Lefschetz $\mathfrak{sl}(2)$-action

COROLLARY: The operators L, Λ, H form a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2)$, with relations

$$[L,\Lambda] = H, \quad [H,L] = 2L, \quad [H,\Lambda] = -2\Lambda.$$

DEFINITION: L, Λ, H is called **the Lefschetz $\mathfrak{sl}(2)$-triple**.

REMARK: **Finite-dimensional representations of $\mathfrak{sl}(2)$ are semisimple**.

REMARK: A simple finite-dimensional representation V of $\mathfrak{sl}(2)$ is generated by $v \in V$ which satisfies $\Lambda(v) = 0$, $H(v) = pv$ **("lowest weight vector")**, where $p \in \mathbb{Z}^{\geqslant 0}$. Then $v, L(v), L^2(v), ..., L^p(v)$ form a basis of $V_p := V$. **This representation is determined uniquely by p.**

REMARK: In this basis, H **acts diagonally:** $H(L^i(v)) = (2i-p)L^i(v)$.

REMARK: One has $V_p = \operatorname{Sym}^p V_1$, where V_1 is a 2-dimensional tautological representation. It is called **a weight p representation of $\mathfrak{sl}(2)$**.

COROLLARY: For a finite-dimensional representation V of $\mathfrak{sl}(2)$, denote by $V^{(i)}$ the eigenspaces of H, with $H\Big|_{V^{(i)}} = i$. **Then L^i induces an isomorphism $V^{(-i)} \xrightarrow{L^i} V^{(i)}$ for any $i > 0$.**

Integrability of the complex structure

CLAIM: ("Cartan's formula") The de Rham differential of can be expressed through the commutator of vector fields:

$$d\eta(X_1, ...X_{d+1}) = \sum (-1)^{i+1} D_{X_i}(\eta(X_1, ..., \check{X}_i, ..., X_{d+1}) \\ - \sum_{i<j} (-1)^{i+j+1} \eta([X_i, X_j], X_1, ..., \check{X}_i, ..., \check{X}_j, ..., X_{d+1}).$$

For a 1-form η, this gives $d\eta(X_1, X_2) = D_{X_1}\eta(X_2) - D_{X_2}\eta(X_1) - \eta([X_1, X_2])$.

COROLLARY: Let (M, I) be an almost complex manifold. Then the following assertions are equivalent.

(i) $d\eta \subset \Lambda^{0,2}(M) \oplus \Lambda^{1,1}(M)$ for any $\eta \in \Lambda^{0,1}(M)$.

(ii) I **is integrable.**

REMARK: This is equivalent to $d\big|_{\Lambda^1 M}$ having only two Hodge components: $d = d^{1,0} + d^{0,1}$ (for a non-integrable complex structure, there are 4: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$).

REMARK: Since $\Lambda^* M$ is multiplicatively generated by $\Lambda^1(M)$, **the decomposition $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$ holds for any almost complex manifold.**

Integrability and the Hodge decomposition

CLAIM: A manifold (M, I) is integrable if and only if $(d^{0,1})^2|_{C^\infty M} = 0$.

Proof. Step 1: The bundle $\Lambda^{1,0}(M)$ is generated over $C^\infty M$ by $d^{1,0}(C^\infty M)$. Indeed, it is n-dimensional, $n = \dim_{\mathbb{C}} M$ and to prove this one needs to find n functions $f_1, ..., f_n$ with $d^{1,0} f_i$ linearly independent at a point. This is done by taking $2n$ functions $f_1, ..., f_{2n}$ with df_i linearly independent, and finding an appropriate subset.

Step 2: Then, the integrability condition $d(\Lambda^{1,0}(M)) \subset \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M)$ is equivalent to $dd^{1,1}(C^\infty M) \subset \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M) \Leftrightarrow d^{-1,2}(d^{1,0}(C^\infty M)) = 0$.

Step 3: The $(0,2)$ component of $d^2 = 0$ gives $\{d^{-1,2}, d^{1,0}\} = \{d^{0,1}, d^{0,1}\} = 2(d^{0,1})^2 = 0$. From Step 2, we obtain that $(d^{0,1})^2|_{C^\infty M} = 0$ **is equivalent to integrability.** ■

REMARK: $d^{2,-1} : \Lambda^{0,1} M \longrightarrow \Lambda^{2,0} M)$ is a $C^\infty M$-linear map which is **dual to the Nijenhuis tensor** $N : \Lambda^2 T^{1,0} M \longrightarrow T^{0,1} M$.

REMARK: The above claim provides an equivalence $d^{2,-1} = 0 \Leftrightarrow \{d^{-1,2}, d^{1,0}\} = 0 \Leftrightarrow (d^{0,1})^2 = 0$.

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := -IdI$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\, d^c}{2}$, $\overline{\partial} := \frac{d-\sqrt{-1}\, d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\overline{\partial} = d^{0,1}$.

Proof: Let V be a space generated by d, IdI. The natural action of $U(1)$ generated by $e^{\mathcal{W}}$ preserves V. **Since d has only two Hodge components. $U(1)$ acts with weights $\sqrt{-1}$ and $-\sqrt{-1}$,** and its Hodge components are expressed as above. ∎

CLAIM: On a complex manifold, one has $d^c = [\mathcal{W}, d]$.

Proof: Clearly, $[\mathcal{W}, d^{1,0}] = \sqrt{-1}\, d^{1,0}$ and $[\mathcal{W}, d^{0,1}] = -\sqrt{-1}\, d^{0,1}$. Adding these equations, obtain $d^c = [\mathcal{W}, d]$.

COROLLARY: $\{d, d^c\} = \{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1).

De Rham differential on Kaehler manifolds

THEOREM: The following statements are equivalent.

1. I is integrable. 2. $\partial^2 = 0$. 3. $\overline{\partial}^2 = 0$. 4. $dd^c = -d^c d$ 5. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

THEOREM: Let M be a Kaehler manifold. One has the following identities **("Kodaira idenitities").**

$$[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [L, \overline{\partial}] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \overline{\partial}^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\overline{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \qquad [L, d^*] = -d^c, \qquad [\Lambda, d^c] = -d^*, \qquad [L, (d^c)^*] = d.$$

Laplacians and supercommutators

THEOREM: Let

$$\Delta_d := \{d, d^*\}, \quad \Delta_{d^c} := \{d^c, d^{c*}\}, \quad \Delta_\partial := \{\partial, \partial^*\}, \Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\}.$$

Then $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. In particular, Δ_d **preserves the Hodge decomposition.**

Proof: By Kodaira relations, $\{d, d^c\} = 0$. Graded Jacobi identity gives

$$\{d, d^*\} = -\{d, \{\Lambda, d^c\}\} = \{\{\Lambda, d\}, d^c\} = \{d^c, d^{c*}\}.$$

Same calculation with $\partial, \overline{\partial}$ gives $\Delta_\partial = \Delta_{\overline{\partial}}$.. Also, $\{\partial, \overline{\partial}^*\} = \sqrt{-1}\,\{\partial, \{\Lambda, \partial\}\} = 0$, (Lemma 1), and the same argument implies that **all anticommutators $\partial, \overline{\partial}^*$, etc. all vanish except $\{\partial, \partial^*\}$ and $\{\overline{\partial}, \overline{\partial}^*\}$.** This gives $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$.
■

DEFINITION: The operator $\Delta := \Delta_d$ is called **the Laplacian**.

REMARK: We have proved that **operators $L, \Lambda, d, \mathcal{W}$ generate a Lie superalgebra of dimension $(5|4)$ (5 even, 4 odd), with a 1-dimensional center $\mathbb{R}\Delta$.**