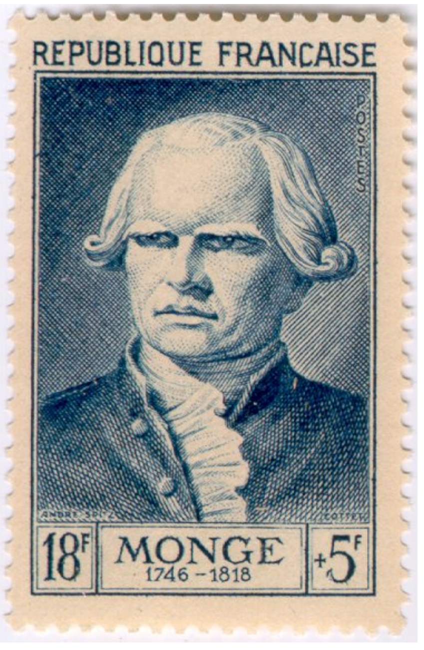
# History of Monge-Ampère equation

Misha Verbitsky

**National Meet on History of Mathematical Sciences**
**January 7, 2010.**
**University of Delhi**

## The Monge-Ampere equation in dimension 2




Gaspard Monge, Comte de Péluse

(10 May 1746 - 28 July 1818)

André-Marie Ampère

(20 January 1775 - 10 June 1836)

$$L[u] = A(u_{xx}u_{yy} - u_{xy}^2) + Bu_{xx} + Cu_{xy} + Du_{yy} + E = 0$$

Monge, G., Sur le calcul intégral des équations aux differences partielles, Mémoires de l'Académie des Sciences, 1784.

Amp‘ere, A.M., Mémoire contenant l'application de la théorie, Journal de l'Ecole Polytechnique, 1820.

## Real Monge-Ampère equation

**DEFINITION:** Let $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a twice differentiable function, $\frac{d^2\varphi}{dx_i dx_j}$ its Hessian matrix. **The real Monge-Ampère equation** is given by

$$\det\left(\frac{d^2\varphi}{dx_i dx_j} - A(x,\varphi,d\varphi)\right) = F(x,\varphi,d\varphi),$$

where $\varphi$ is unknown, and $F$ a given function.

**REMARK: It is elliptic, if $\varphi$ is convex, and the matrix $A$ is positive definite.**

**REMARK:** Let $\rho: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, $M$ its graph, and $K$ the Gaussian curvature of $M$, considered as a function of $\mathbb{R}^n$. Then

$$K(x) = \frac{\det\left(\frac{d^2\rho}{dx_i dx_j}\right)}{(1+|d\rho|^2)^{(n+2)/2}}.$$

**To find a surface with a prescribed Gaussian curvature, one has to solve the Monge-Ampère equation**

$$\det\left(\frac{d^2\rho}{dx_i dx_j}\right) = K(x)(1+|d\rho|^2)^{(n+2)/2}.$$

## The Monge optimal transportation problem


Gaspard Monge, Comte de Péluse
(10 May 1746 - 28 July 1818)

Let $\Omega, \Omega'$ be domains in $\mathbb{R}^n$, $c: \Omega \times \Omega' \longrightarrow \mathbb{R}$ **a "cost function"** (expressing the cost of transportation from a point of $\Omega$ to $\Omega'$) and $f, g$ measures on $\Omega$, $\Omega'$ satisfying $\int_\Omega f = \int_{\Omega'} g$. For a measure-preserving **transportation function** $T: \Omega \longrightarrow \Omega'$, consider its cost functional

$$C(T) := \int_\Omega c(x, T(y))$$

**The Problem** (Monge, 1784): **Find a transportation function which minimizes the cost.**

## Transport Monge-Ampère Equation

**Solution** A cost-minimizing function $T$ satisfies $T = \nabla\varphi$, for some convex function $\varphi$ on $\Omega$. Moreover,

$$\det(\mathrm{Hess}(\varphi) - c_{xx}(x, T(x))) = \frac{f(x)}{g(T(x))}$$

where $\mathrm{Hess}(\varphi) := \frac{d^2\varphi}{dx_i dx_j}$

**DEFINITION:** This equation is called **The transport Monge-Ampère equation**

**REMARK: Still studied in applied math and economics** ("Kantorovich-Monge", "Monge-Ampere-Kantorovich").

**REMARK:** In Monge's paper, the cost function is $c(x, y) = |x - y|$. **Uniqueness of solutions was obtained only recently** (Sudakov, Trudinger-Wang, Caffarelli-Feldman-McCann)

## Solving the Monge-Ampère Equation

1. **Uniqueness of solutions** (on compacts or with prescribed boundary conditions).

2. **Existence of weak solutions** (solutions which are geberalized functions, that is, with singularities).

3. **Elliptic regularity** (every weak solution is in fact smooth and real analytic).

## Continuity method of S.-T. Yau.

0. Suppose we have a Monge-Ampere equation $MA(\varphi) = F_t$ depending from $t \in [0, 1]$. Solve $MA(\varphi) = F_t$ for $t = 0$. **Prove that the set of $t$ for which one can solve $MA(\varphi) = F_t$ is open and closed.**

1. Let $C \subset [0, 1]$ be the set of all $t$ for which $MA(\varphi) = F_t$ has a solution. Prove that $C$ is open (straightforward, because MA is elliptic).

2. A limit of solutions of $MA(\varphi) = F_t$ is a weak solution.

3. Using **a priori estimates**, prove that a weak solution is regular.

## Complex Monge-Ampère Equation

**DEFINITION:** Let $\varphi$ be a function on $\mathbb{C}^n$, and $dd^c\varphi$ its **complex Hessian**, $dd^c\varphi := \mathrm{Hess}(\varphi) + I(\mathrm{Hess}(\varphi))$. It is a Hermitian form.

**CLAIM:** The form $dd^c\varphi$ is **independent from the choice of complex coordinates**.

**REMARK:** The usual (real) Hessian is much less invariant.

**DEFINITION:** A **Kaehler manifold** is a complex manifold with a Hermitian metric $g$ which is locally represented as $g = dd^c\psi$.

**DEFINITION:** Let $(M, g)$ be a Kaehler manifold. **The complex Monge-Ampere equation is**

$$\det(g + dd^c\varphi) = e^f$$

**THEOREM:** (Yau) On a compact Kaehler manifold, **the complex Monge-Ampere equation has a unique solution,** for any smooth function $f$ subject to constraint $\int_M e^f \mathrm{Vol}_g = \int_M \mathrm{Vol}_g$.

## Calabi-Yau manifolds

**DEFINITION:** A compact Kähler manifold $(M,g)$, $\dim_{\mathbb{C}} M = n$ is called **a Calabi-Yau manifold** if $M$ admits a non-degenerate $(n,0)$-differential form, equivalently, if $c_1(M) = 0$.

**DEFINITION:** The Levi-Civita connection on $TM$ induces a connection on the bundle $\Lambda^{n,0}(M)$ of **holomorphic volume forms**. Its curvature is called **the Ricci curvature of $M$**.

**REMARK: Let $\Phi \in \Lambda^{n,0}(M)$. Then $\operatorname{Ric}(M) = dd^c \log |\Phi|^2$.**

**REMARK:** Let $g, g'$ be Hermitian metrics on $M$. Then

$$\frac{|\Phi|^2_{g'}}{|\Phi|^2_g} = \frac{\det g'}{\det g}$$

In particular, **a Kaehler metric $g'$ is Ricci-flat if and only if $\det g' = \frac{|\Phi|^2_g}{\det g}$.**

**THEOREM:** (Calabi-Yau) **Every Calabi-Yau manifold admits a Ricci-flat Kähler metric.**

**Proof:** Solve the Monge-Ampere equation $\det g' = \frac{|\Phi|^2_g}{\det g}$. ∎

## Applications of Calabi-Yau theorem

1. **Deformations of Calabi-Yau manifolds are unobstructed** (Bogomolov-Tian-Todorov). Applications to Mirror Symmetry.

2. **Global Torelli theorem for holomorphically symplectic manifolds** (in particular, a K3 surface). Classification of surfaces.

3. **Existence of Kaehler currents (limits of Kaehler metrics) with prescribed singularities** (Demailly-Paun). Characterization of Kaehler classes and manifolds of Fujiki class C.

4. Kaehler metrics in a given Kaehler class are parametrized by their volumes.

## Calabi-Yau theorem for real Monge-Ampère equation

**DEFINITION:** A manifold with flat torsion-free connection is called **an affine manifold**.

**DEFINITION:** A metric $g$ on an affine manifold is called **a Hessian metric** if locally it can be written as $g = \operatorname{Hess}(\varphi)$, for some convex function $\varphi$.

**THEOREM:** (Cheng-Yau) Let $(M, g)$ be a compact affine manifold with a Hessian metric. Assume that the flat connection $\nabla$ preserves a volume form $V$. Let $f$ be a function on $M$ which satisfies $\int_M V = \int_M e^f V$. **Then the equation**

$$\det(g + \operatorname{Hess}(\varphi)) = e^f V$$

**has a unique smooth solution $\varphi$.** ■

**REMARK:** There is an earlier theorem of Pogorelov, who proved that **on $\mathbb{R}^n$ any convex solution of $\operatorname{Hess}(\varphi) = const$ is quadratic.**

## Hypercomplex manifolds

**Definition:** Let $M$ be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that $I$, $J$, $K$ are integrable almost complex structures. Then

$$(M, I, J, K)$$

is called **a hypercomplex manifold**.

**REMARK:** Calabi-Yau theorem implies that **every holomorphically symplectic manifold admits a hypercomplex structure.**

**DEFINITION:** Let $(M, I, J, K)$ be a hypercomplex manifold, and $g$ a Riemannian metric. We say that $g$ is **quaternionic Hermitian** if $I, J, K$ are orthogonal with respect to $g$.

## Quaternionic Monge-Ampere equation

**CLAIM:** Let $g$ be any metric, and $g_{SU(2)} := g + I(g) + J(g) + K(g)$. **Then $g$ is quaternionic Hermitian.**

**DEFINITION:** Let $M$ be a hypercomplex manifold. **A quaternionic Hessian** of a function $\varphi$ is

$$\operatorname{Hess}_{\mathbb{H}}(\varphi) := \operatorname{Hess}(\varphi) + I \operatorname{Hess}(\varphi) + J \operatorname{Hess}(\varphi) + K \operatorname{Hess}(\varphi).$$

**DEFINITION: An HKT metric** on a hypercomplex manifold is a quaternionic Hermitian metric which is locally a quaternionic Hessian of a function.

**CONJECTURE:** (quaternionic Monge-Ampere equation) Let $M$ be a compact hypercomplex manifold, $\dim_{\mathbb{H}} M = n$, and $g$ its HKT-metric. Assume that $(M, I)$ admits a nowhere degenerate holomorphic $(2n, 0)$-form $\Phi$, and let $f$ be a function which satisfies

$$\int_M e^f \Phi \wedge \overline{\Phi} = \int_M \Phi \wedge \overline{\Phi}.$$

**Then the quaternionic Monge-Ampere equation**

$$\det(g + \operatorname{Hess}_{\mathbb{H}}(\varphi)) = e^f \Phi \wedge \overline{\Phi}$$

**has a unique solution $\varphi$.**