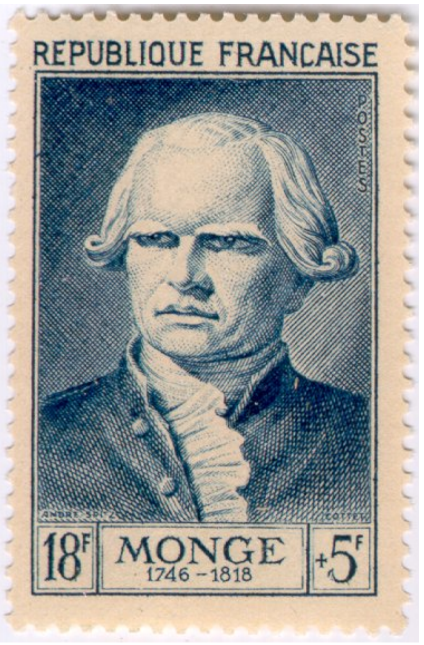
History of Monge-Ampère equation

Misha Verbitsky

National Meet on History of Mathematical Sciences
January 7, 2010.
University of Delhi

The Monge-Ampere equation in dimension 2



Gaspard Monge, Comte de Péluse

(10 May 1746 - 28 July 1818)

André-Marie Ampère

(20 January 1775 - 10 June 1836)

$$L[u] = A(u_{xx}u_{yy} - u_{xy}^2) + Bu_{xx} + Cu_{xy} + Du_{yy} + E = 0$$

Monge, G., Sur le calcul intégral des équations aux differences partielles, Mémoires de l'Académie des Sciences, 1784.

Amp'ere, A.M., Mémoire contenant l'application de la théorie, Journal de l'Ecole Polytechnique, 1820.

Real Monge-Ampère equation

DEFINITION: Let $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a twice differentiable function, $\frac{d^2\varphi}{dx_i dx_j}$ its Hessian matrix. **The real Monge-Ampère equation** is given by

$$\det\left(\frac{d^2\varphi}{dx_i dx_j} - A(x,\varphi,d\varphi)\right) = F(x,\varphi,d\varphi),$$

where φ is unknown, and F a given function.

REMARK: It is elliptic, if φ is convex, and the matrix A is positive definite.

REMARK: Let $\rho: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, M its graph, and K the Gaussian curvature of M, considered as a function of $\mathbb{R}^n$. Then

$$K(x) = \frac{\det\left(\frac{d^2\rho}{dx_i dx_j}\right)}{(1+|d\rho|^2)^{(n+2)/2}}.$$

To find a surface with a prescribed Gaussian curvature, one has to solve the Monge-Ampère equation

$$\det\left(\frac{d^2\rho}{dx_i dx_j}\right) = K(x)(1+|d\rho|^2)^{(n+2)/2}.$$

The Monge optimal transportation problem

Gaspard Monge, Comte de Péluse
(10 May 1746 - 28 July 1818)

Let Ω, Ω' be domains in $\mathbb{R}^n$, $c: \ \Omega \times \Omega' \longrightarrow \mathbb{R}$ **a "cost function"** (expressing the cost of transportation from a point of Ω to Ω') and f, g measures on Ω, Ω' satisfying $\int_\Omega f = \int_{\Omega'} g$. For a measure-preserving **transportation function** $T: \ \Omega \longrightarrow \Omega'$, consider its cost functional

$$C(T) := \int_\Omega c(x, T(y))$$

The Problem (Monge, 1784): **Find a transportation function which minimizes the cost.**

Transport Monge-Ampère Equation

Solution A cost-minimizing function T satisfies $T = \nabla\varphi$, for some convex function φ on Ω. Moreover,

$$\det(\mathrm{Hess}(\varphi) - c_{xx}(x, T(x))) = \frac{f(x)}{g(T(x))}$$

where $\mathrm{Hess}(\varphi) := \frac{d^2\varphi}{dx_i dx_j}$

DEFINITION: This equation is called **The transport Monge-Ampère equation**

REMARK: Still studied in applied math and economics ("Kantorovich-Monge", "Monge-Ampere-Kantorovich").

REMARK: In Monge's paper, the cost function is $c(x, y) = |x - y|$. **Uniqueness of solutions was obtained only recently** (Sudakov, Trudinger-Wang, Caffarelli-Feldman-McCann)

Solving the Monge-Ampère Equation

1. **Uniqueness of solutions** (on compacts or with prescribed boundary conditions).

2. **Existence of weak solutions** (solutions which are geberalized functions, that is, with singularities).

3. **Elliptic regularity** (every weak solution is in fact smooth and real analytic).

Continuity method of S.-T. Yau.

0. Suppose we have a Monge-Ampere equation $MA(\varphi) = F_t$ depending from $t \in [0,1]$. Solve $MA(\varphi) = F_t$ for $t = 0$. **Prove that the set of t for which one can solve $MA(\varphi) = F_t$ is open and closed.**

1. Let $C \subset [0,1]$ be the set of all t for which $MA(\varphi) = F_t$ has a solution. Prove that C is open (straightforward, because MA is elliptic).

2. A limit of solutions of $MA(\varphi) = F_t$ is a weak solution.

3. Using **a priori estimates**, prove that a weak solution is regular.

Complex Monge-Ampère Equation

DEFINITION: Let φ be a function on $\mathbb{C}^n$, and $dd^c\varphi$ its **complex Hessian**, $dd^c\varphi := \mathrm{Hess}(\varphi) + I(\mathrm{Hess}(\varphi))$. It is a Hermitian form.

CLAIM: The form $dd^c\varphi$ is **independent from the choice of complex coordinates**.

REMARK: The usual (real) Hessian is much less invariant.

DEFINITION: A **Kaehler manifold** is a complex manifold with a Hermitian metric g which is locally represented as $g = dd^c\psi$.

DEFINITION: Let (M, g) be a Kaehler manifold. **The complex Monge-Ampere equation is**

$$\det(g + dd^c\varphi) = e^f$$

THEOREM: (Yau) On a compact Kaehler manifold, **the complex Monge-Ampere equation has a unique solution,** for any smooth function f subject to constraint $\int_M e^f \mathrm{Vol}_g = \int_M \mathrm{Vol}_g$.

Calabi-Yau manifolds

DEFINITION: A compact Kähler manifold (M, g), $\dim_{\mathbb{C}} M = n$ is called **a Calabi-Yau manifold** if M admits a non-degenerate $(n, 0)$-differential form, equivalently, if $c_1(M) = 0$.

DEFINITION: The Levi-Civita connection on TM induces a connection on the bundle $\Lambda^{n,0}(M)$ of **holomorphic volume forms**. Its curvature is called **the Ricci curvature of M**.

REMARK: Let $\Phi \in \Lambda^{n,0}(M)$. Then $\operatorname{Ric}(M) = dd^c \log |\Phi|^2$.

REMARK: Let g, g' be Hermitian metrics on M. Then

$$\frac{|\Phi|^2_{g'}}{|\Phi|^2_g} = \frac{\det g'}{\det g}$$

In particular, **a Kaehler metric g' is Ricci-flat if and only if $\det g' = \frac{|\Phi|^2_g}{\det g}$.**

THEOREM: (Calabi-Yau) **Every Calabi-Yau manifold admits a Ricci-flat Kähler metric.**

Proof: Solve the Monge-Ampere equation $\det g' = \frac{|\Phi|^2_g}{\det g}$. ∎

Applications of Calabi-Yau theorem

1. **Deformations of Calabi-Yau manifolds are unobstructed** (Bogomolov-Tian-Todorov). Applications to Mirror Symmetry.

2. **Global Torelli theorem for holomorphically symplectic manifolds** (in particular, a K3 surface). Classification of surfaces.

3. **Existence of Kaehler currents (limits of Kaehler metrics) with prescribed singularities** (Demailly-Paun). Characterization of Kaehler classes and manifolds of Fujiki class C.

4. Kaehler metrics in a given Kaehler class are parametrized by their volumes.

Calabi-Yau theorem for real Monge-Ampère equation

DEFINITION: A manifold with flat torsion-free connection is called **an affine manifold**.

DEFINITION: A metric g on an affine manifold is called **a Hessian metric** if locally it can be written as $g = \text{Hess}(\varphi)$, for some convex function φ.

THEOREM: (Cheng-Yau) Let (M, g) be a compact affine manifold with a Hessian metric. Assume that the flat connection ∇ preserves a volume form V. Let f be a function on M which satisfies $\int_M V = \int_M e^f V$. **Then the equation**

$$\det(g + \text{Hess}(\varphi)) = e^f V$$

has a unique smooth solution φ. ■

REMARK: There is an earlier theorem of Pogorelov, who proved that **on $\mathbb{R}^n$ any convex solution of $\text{Hess}(\varphi) = const$ is quadratic.**

Hypercomplex manifolds

Definition: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}.$$

Suppose that I, J, K are integrable almost complex structures. Then

$$(M, I, J, K)$$

is called **a hypercomplex manifold**.

REMARK: Calabi-Yau theorem implies that **every holomorphically symplectic manifold admits a hypercomplex structure.**

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I, J, K are orthogonal with respect to g.

Quaternionic Monge-Ampere equation

CLAIM: Let g be any metric, and $g_{SU(2)} := g + I(g) + J(g) + K(g)$. **Then g is quaternionic Hermitian.**

DEFINITION: Let M be a hypercomplex manifold. **A quaternionic Hessian** of a function φ is

$$\operatorname{Hess}_{\mathbb{H}}(\varphi) := \operatorname{Hess}(\varphi) + I \operatorname{Hess}(\varphi) + J \operatorname{Hess}(\varphi) + K \operatorname{Hess}(\varphi).$$

DEFINITION: **An HKT metric** on a hypercomplex manifold is a quaternionic Hermitian metric which is locally a quaternionic Hessian of a function.

CONJECTURE: (quaternionic Monge-Ampere equation) Let M be a compact hypercomplex manifold, $\dim_{\mathbb{H}} M = n$, and g its HKT-metric. Assume that (M, I) admits a nowhere degenerate holomorphic $(2n, 0)$-form Φ, and let f be a function which satisfies

$$\int_M e^f \Phi \wedge \overline{\Phi} = \int_M \Phi \wedge \overline{\Phi}.$$

Then the quaternionic Monge-Ampere equation

$$\det(g + \operatorname{Hess}_{\mathbb{H}}(\varphi)) = e^f \Phi \wedge \overline{\Phi}$$

has a unique solution φ.