Algebraic geometry over quaternions

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January 27, 2010,

Chandigarh

Quaternionic geometry: an introduction

DEFINITION: An isometry is a map which preserve distances. **Euclidean geometry is a study of isometries (Felix Klein).**

Isometries of $\mathbb{R}^2$ are expressed in terms of complex numbers. Translations correspond to addition, turns to multiplication. An isometry of a plane can be written as a map of complex numbers $z \longrightarrow az + b$, where a, b are complex numbers, $|a| = 1$.

This allows one to answer geometry questions algebraically.

QUESTION: Can we do that in dimension 3?

ANSWER: Yes!

Sir William Rowan Hamilton
(August 4, 1805 – September 2, 1865)

Broom Bridge

"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication

$$I^2 = J^2 = K^2 = IJK = -1$$

and cut it on a stone of this bridge."

Isometries in $\mathbb{R}^3$.

DEFINITION: Quaternions are the algebra of **"quaternion numbers"** $\mathbb{H} = \langle aI + bJ + cK + d$, with $a, b, c, d, \in \mathbb{R}$ (real numbers), and relations $I^2 = J^2 = K^2 = IJK = -1$.

DEFINITION: Define **the conjugate** quaternion to be $\overline{aI + bJ + cK + d} = -aI - bJ - cK + d$ and **the norm** of a quaternion $|h| := \sqrt{h\overline{h}}$.

REMARK: $\overline{xy} = \overline{y}\overline{x}$.

REMARK: $|aI + bJ + cK + d| = \sqrt{a^2 + b^2 + c^2 + d^2}$.

REMARK: The norm is multiplicative (preserves multiplication of quaternions): $|xy|^2 = xy\overline{y}\overline{x} = x|y|^2\overline{x} = |x|^2|y|^2$.

We identify $\mathbb{R}^3$ with the space of imaginary quaternions, $\mathbb{R}^3 = aI + bJ + cK$, and define an action of $SU(2)$ on $\mathbb{R}^3$ by the formula $h(v) = hvh^{-1}$

REMARK: This is an isometry! Indeed, $|h(v)| = |h||v||h|^{-1}$.

REMARK: Any isometry of $\mathbb{R}^3$ can be written as $v \longrightarrow hvh^{-1} + p$, where $h \in \mathbb{H}$ and $p \in \mathbb{R}^3$.

Isometries in $\mathbb{R}^4$.

REMARK: A group is a set equipped with an associative multiplication, which is invertible, and a unit.

DEFINITION: The group of **unitary quaternions** $h \in H, |h| = 1$ is called $SU(2)$.

Define an action of $SU(2) \times SU(2)$ on $\mathbb{R}^4 = \mathbb{H}$: $\rho(h_1, h_2)(v) = h_1 v h_2^{-1}$. **This is an isometry!**

CLAIM: Every isometry of R^4 can be written as as $v \longrightarrow h_1 v h_2^{-1} + p$, for appropriate $h_1, h_2 \in SU(2)$ and $p \in \mathbb{H}$.

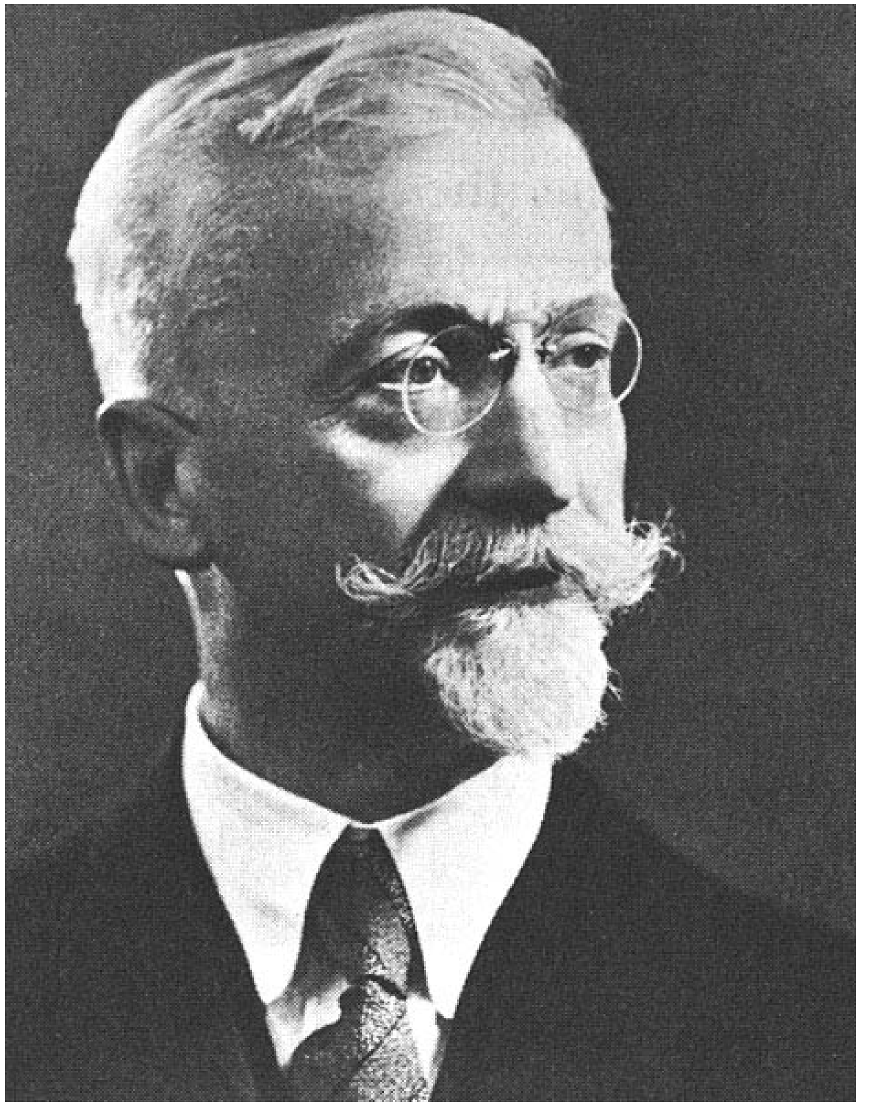
REMARK: A **group isomorphism** $G \cong G'$ is a one-to-one correspondence between the groups G, G' which is multiplicative.

REMARK: The correspondence observed by Hamilton can be written in the modern language as $SU(2)/\pm 1 \cong SO(3)$, where $SO(3)$ is the group of isometries of $\mathbb{R}^3$ preserving 0.

REMARK: For $\mathbb{R}^4$, one also has an isomorphism $SO(4) = SU(2) \times SU(2)/\pm 1$.

This is called **the spin covering**.

Fast forward 70 years.



Élie Joseph Cartan
(9 April 1869 – 6 May 1951)

Geometric structures

DEFINITION: A geometric structure (Elie Cartan) is an atlas on a manifold, with the differentials of all transition functions in a given subgroup $G \subset GL(n, \mathbb{R})$.

EXAMPLE: $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ ("the complex structure").

EXAMPLE: $Sp(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ ("the symplectic structure").

"Quaternionic structures" in the sense of Elie Cartan **don't exist.**

THEOREM: Let $f: \mathbb{H}^n \longrightarrow \mathbb{H}^m$ be a function, defined locally in some open subset of n-dimensional quaternion space $\mathbb{H}^n$. Suppose that the differential Df is $\mathbb{H}$-linear. **Then f is a linear map.**

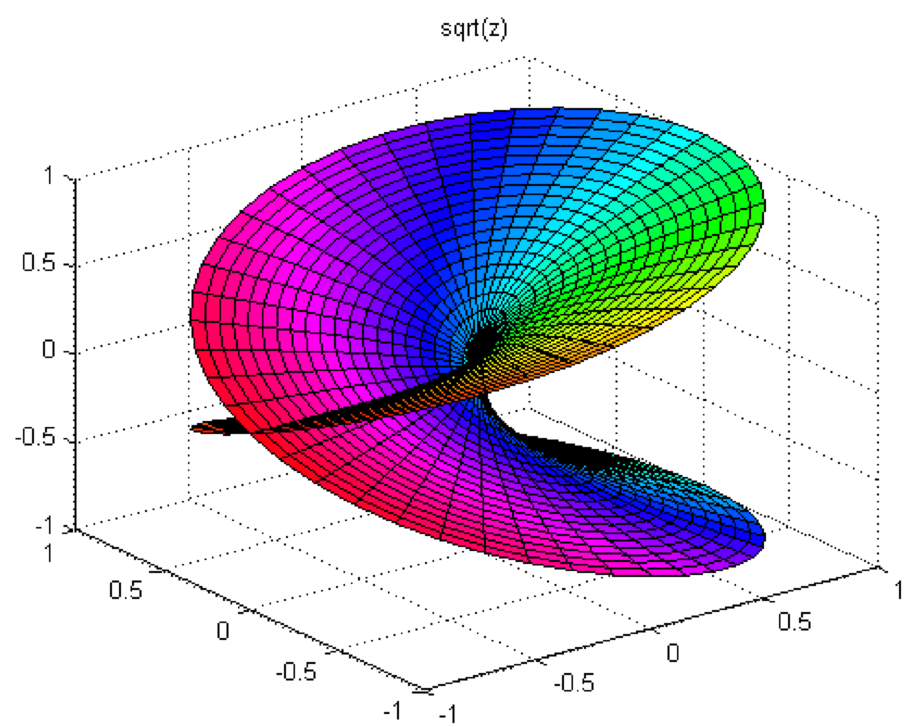
Proof (a modern one): The graph of f is a "hyperkähler submanifold" in $\mathbb{H}^n \times \mathbb{H}^m$, hence "geodesically complete", hence linear. ■

Algebraic geometry over $\mathbb{C}$ is a respectable subject. Algebraic geometry over $\mathbb{R}$ as well (maybe a bit less respectable, but anyway).

Is there algebraic geometry over $\mathbb{H}$?

History of algebraic geometry.

1. XIX centrury: Riemann, Klein, Poincaré. Study of elliptic integrals and elliptic functions leads to the notion of a **Riemannian surface** of a holomorphic function. In a modern language, Riemann surface is a smooth 2-dimensional manifold, covered by open disks in $\mathbb{R}^2 = \mathbb{C}$, with transition functions holomorphic.



A Riemann surface for a square root.

History of algebraic geometry.

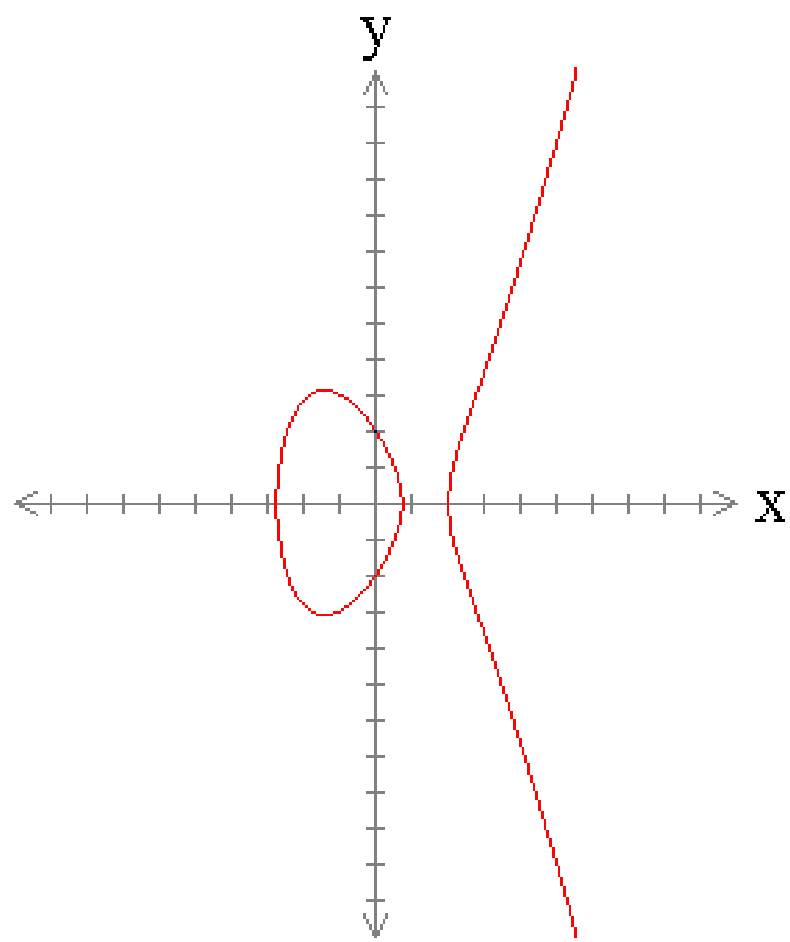
2. Italian school (1885-1935):
Segre, Severi, Enriques, Castelnuovo.

An **affine algebraic variety** is a subset in $\mathbb{C}^n$ defined as a set of common zeroes of a system of algebraic equations. Two varieties are equivalent, if there exists a polynomial bijection from one to another.

1. **Can be defined over any algebraically closed field.**

2. If the equations are homogeneous, they define a (compact) subset in a projective space $\mathbb{C}P^n$ (**"a projective variety"**)

3. Definition is not intrinsic.



$y^2 = x^3 - 6x + 4$

An elliptic curve

3. Modern approach: Zariski, Weil, Grothendieck, Dieudonné

A **scheme** is a ringed space which is locally isomorphic to a spectrum of a ring (with Zariski topology). Morphisms of schemes are morphisms of ringed spaces: continuous maps $X \xrightarrow{\varphi} Y$, with ring homomorphisms

$$\varphi^* : \mathcal{O}_U \longrightarrow \mathcal{O}_{\varphi^{-1}(U)}$$

defined for any open $U \subset Y$ and commuting with restrictions to subsets.

0. Scheme geometry. All the usual geometric notions (compactness, separability, smoothness...) have their scheme-theoretic versions.

1. Schemes are closed under all natural operations.
(taking products, a graph of a morphism, intersection, union...)

2. The moduli spaces are again schemes (when finite-dimensional).
The **moduli spaces** are the sets parameterizing various algebro-geometric objects (subvarieties, morphisms, fiber bundles) and equipped with a natural algebraic structure. Grothendieck proved that **the moduli exist in scheme category, under very general assumptions.**

3. Can be used in number theory.
The rings do not need to be defined over $\mathbb{C}$, or any other algebraically closed field. In particular, $\mathrm{Spec}(\mathbb{Z})$ **is a scheme**, which can be studied in geometric terms. This was the original motivation of Grothendieck (at least, one of his motivations).

4. Desingularization (Hironaka).
Over a field of characteristic 0, **any variety X admits a desingularization**, that is, a proper, surjective map $\tilde{X} \longrightarrow X$, with $\tilde{X}$ smooth, and generically one-to-one.

Complex geometry (Grauert, Oka, Cartan, Serre...)

DEFINITION: A **complex manifold** is a manifold with an atlas of open subsets in $\mathbb{C}^n$, and translation maps complex analytic.

DEFINITION: A **complex analytic subvariety** is a closed subset, locally defined as a zero set of a system of complex analytic equations. A **complex analytic variety** is a ringed topological space, locally isomorphic to a closed subvariety of an open ball $B \subset \mathbb{C}^n$.

Complex spaces are as good as schemes: **the products/graphs/moduli spaces of complex spaces are again complex spaces,** and Hironaka's desingularization works as well.

REMARK: Since any complex algebraic map is complex analytic, **every scheme defines a complex analytic space**.

DEFINITION: A complex variety obtained from a scheme is called **algebraic**.

Serre's GAGA (Géométrie Algébrique - Géométrie Analitique, 1956): A complex subvariety of a compact algebraic variety is algebraic. **Compact algebraic varieties over $\mathbb{C}$ are special case of complex analytic!**

However, the topology of complex varieties is **infinitely more complicated.**

Kähler manifolds.

A complex manifold is equipped with a natural map $I\ TM \longrightarrow TM$, $I^2 = -\operatorname{Id}$, called **the complex structure map**. A Riemannian metric is called **Hermitian** if $g(Ix, y) = g(x, Iy)$. In this situation $\omega(x, y) = g(x, Iy)$ is a differential form, called Hermitian form. The following conditions are equivalent

1. $d\omega = 0$.

2. ω **is parallel** (preserved by the Levi-Civita connection), that is, $\nabla\omega = 0$.

3. Flat approximation. At each point M has complex coordinates, such that g is approximated at this point by a standard (flat) Hermitian structure in this coordinates, up to order 2.

If any of these conditions is satisfied, the metric is called **Kähler** (after Erich Kähler, 1938).

NB: Kähler manifolds are symplectic.

Properties of Kähler manifolds.

1. The $U(n+1)$-invariant metric on $\mathbb{C}P^n$ is called **the Fubini-Study metric** (its uniqueness and existence follows easily from $U(n)$-invariance). Since $\mathbb{C}^n$ does not have $U(n)$-invariant 3-forms, Fubini-Study metric is **Kähler**.

2. **A submanifold of a Kähler manifold is again Kähler** (restriction of ω is still closed). Therefore, all algebraic manifolds are Kähler.

3. **Topology of compact Kähler manifolds is tightly controlled** (all rational cohomology operations vanish, etc.) The fundamental group is especially easy to control. **It is conjectured that the isomorphism problem for fundamental groups of compact Kähler manifolds has an algorithmic solution.**

4. By contrast, **any finitely-generated, finitely-presented group can be a fundamental group of a compact complex manifold.** Therefore **the problem of recursively enumerating the fundamental groups cannot be solved.**

5. Topology of complex manifolds is infinitely more complicated!

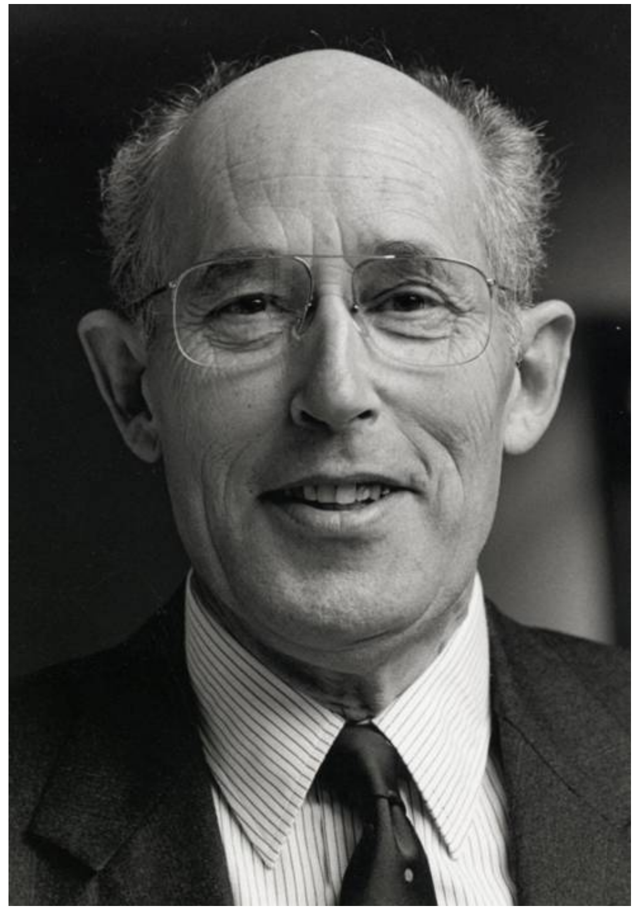
Algebraic geometry over $\mathbb{H}$.

Over $\mathbb{C}$, we have 3 distinct notions of "algebraic geometry":

1. Schemes over $\mathbb{C}$.
2. Complex manifolds.
3. Kähler manifolds.

The first notion does not work for $\mathbb{H}$, because polynomial functions on $\mathbb{H}^n$ generate all real polynomials on $\mathbb{R}^4$. The second version does not work, because any quaternionic-differentiable function is linear. The third one works!

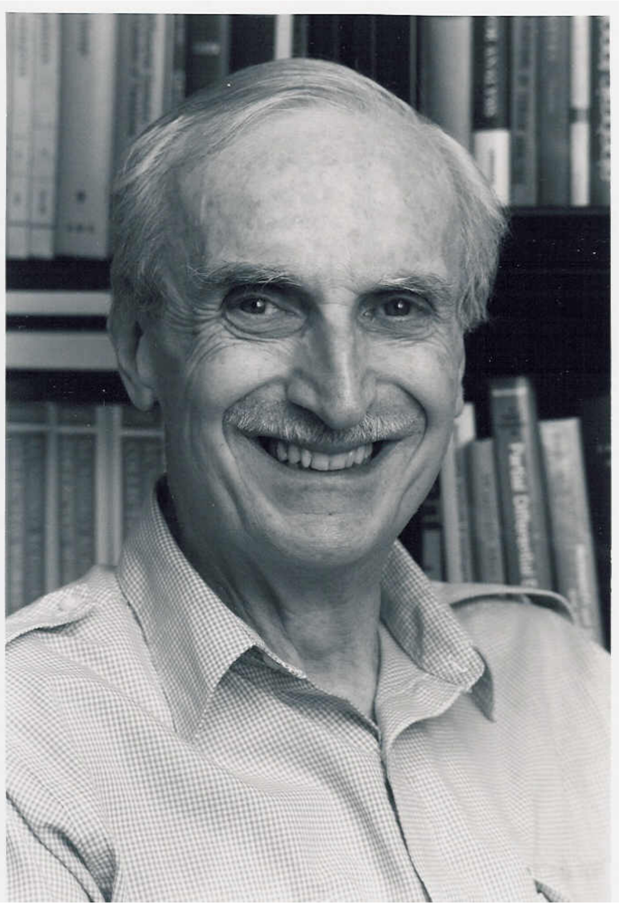
Hyperkähler manifolds.



Marcel Berger

Classification of holonomies.

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds



Eugenio Calabi

DEFINITION: (E. Calabi, 1978)
Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\mathrm{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

Holomorphic symplectic geometry

CLAIM: A hyperkähler manifold (M, I, J, K), considered as a complex manifold (M, I), is **holomorphically symplectic** (equipped with a holomorphic, non-degenerate 2-form). Recall that, M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on (M, I).** ■

Converse is also true, as follows from the famous conjecture, made by Calabi in 1952.

THEOREM: (S.-T. Yau, 1978) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

“Hyperkähler algebraic geometry” is almost as good as the usual one.

Define **trianalytic subvarieties** as closed subsets which are complex analytic with respect to I, J, K.

0. **Trianalytic subvarieties are singular hyperkähler.**

1. Let L be a generic quaternion satisfying $L^2 = -1$. Then **all complex subvarieties of (M, L) are trianalytic.**

2. A normalization of a hyperkähler variety is smooth and hyperkähler. **This gives a desingularization** (“hyperkähler Hironaka”).

3. A complex deformation of a trianalytic subvariety **is again trianalytic**, the corresponding moduli space is (singularly) hyperkähler.

4. Similar results are true for vector bundles which are holomorphic under I, J, K **(“hyperholomorphic bundles”)**