Kähler manifolds and holonomy

lecture 2

Misha Verbitsky

Tel-Aviv University

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Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Levi-Civita connection and Kähler geometry

DEFINITION: Let (M,g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g)=0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

THEOREM: Let (M,I,g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) (M,I,g) **is Kähler**

(ii) One has $\nabla(I)=0$, where ∇ is the Levi-Civita connection.

Holomorphic vector bundles

DEFINITION: A (smooth) **vector bundle** on a smooth manifold is a locally trivial sheaf of $C^\infty M$-modules.

DEFINITION: A **holomorphic vector bundle** on a complex manifold is a locally trivial sheaf of $\mathcal{O}_M$-modules.

REMARK: A section b of a bundle B is often denoted as $b \in B$.

CLAIM: Let B be a holomorphic vector bundle. Consider the sheaf $B_{C^\infty} := B \otimes_{\mathcal{O}_M} C^\infty M$. It is clearly locally trivial, hence **B_{C^∞} is a smooth vector bundle.**

DEFINITION: B_{C^∞} is called **a smooth vector bundle underlying B**.

A holomorphic structure operator

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0}: \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\overline{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMARK: From $d^2 = 0$, one obtains **$\overline{\partial}^2 = 0$ and $\partial^2 = 0$.**

REMARK: The operator $\overline{\partial}$ is $\mathcal{O}_M$-linear.

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: The kernel of $\overline{\partial}$ **coincides with the set of holomorphic sections** of B.

The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: (Atiyah-Bott) Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

REMARK: This statement is a vector bundle analogue of Newlander-Nirenberg theorem.

DEFINITION: $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a connection

DEFINITION: Let $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a smooth budnle. Extend it to an operator on B-valued forms

$$B \xrightarrow{\nabla} \Lambda^1(M) \otimes B \xrightarrow{\nabla} \Lambda^2(M) \otimes B \xrightarrow{\nabla} \Lambda^3(M) \otimes B \xrightarrow{\nabla} ...$$

using $\nabla(\eta \otimes b) = d\eta + (-1)^{\tilde\eta} \eta \wedge \nabla b$. The operator $\nabla^2 : B \longrightarrow B \otimes \Lambda^2(M)$ is called **the curvature** of ∇.

REMARK: The algebra of $\operatorname{End}(B)$-valued forms naturally acts on $\Lambda^* M \otimes B$. The curvature satisfies $\nabla^2(fb) = d^2 fb + df \wedge \nabla b - df \wedge \nabla b + f\nabla^2 b = f\nabla^2 b$, hence it is $C^\infty M$-linear. **We consider it as an $\operatorname{End}(B)$-valued 2-form on M.**

PROPOSITION: (Bianchi identity) Using the **graded Jacobi identity**, we obtain $[\nabla, \nabla^2] = [\nabla^2, \nabla] + [\nabla, \nabla^2] = 0$, hence $[\nabla, \nabla^2] = 0$. This gives **Bianchi identity:** $\nabla(\Theta_B) = 0$.

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B.**

An exercise: Check that $c_1(B)$ is independent from a choice of ∇.

Curvature of a holomorphic line bundle

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0} b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0} b, b) = \operatorname{Re} \eta |b|^2$. **This gives** $\nabla^{1,0} b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.

REMARK: Then $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, **that is,** $\Theta_B = -2\partial\bar\partial \log |b|$.

COROLLARY: If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has** $\Theta' - \Theta = -2\partial\bar\partial f$

CLAIM: Let η be a closed (1,1)-form in the same cohomology class as $\Theta_{B,h}$. **Then η is a curvature of a Chern connection** on B, for some metric h'.

Proof: The difference $\Theta_{B,h} - \eta$ is an exact (1,1)-form, hence **belongs to an image of $\partial\bar\partial$ ("$\partial\bar\partial$-lemma"): $\Theta_{B,h} - \eta = -2\partial\bar\partial f$.** Then the curvature of a metric $h' := e^{2f} h$ satisfies $\Theta_{B,h} - \Theta_{B,h'} = -2\partial\bar\partial f$, hence $\eta = \Theta_{B,h'}$. ■

REMARK: **Such metric is unique, up to a constant.**

Calabi-Yau manifolds

REMARK: Let B be a line bundle on a manifold. Using the long exact sequence of cohomology associated with the exponential sequence

$$0 \longrightarrow \mathbb{Z}_M \longrightarrow C^\infty M \longrightarrow (C^\infty M)^* \longrightarrow 0,$$

we obtain $0 \longrightarrow H^1(M, (C^\infty M)^*) \longrightarrow H^2(M, \mathbb{Z}) \longrightarrow 0$**.**

DEFINITION: Let B be a complex line bundle, and ξ_B its defining element in $H^1(M, (C^\infty M)^*)$. Its image in $H^2(M, \mathbb{Z})$ is called **the integer first Chern class** of B.

REMARK: A complex line bundle B is (topologically) trivial if and only if $c_1(B) = 0$.

THEOREM: (Gauss-Bonnet) A real Chern class of a vector bundle **is an image of the integer Chern class $c_1(B, \mathbb{Z})$** under the natural homomorphism $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{R})$.

DEFINITION: A first Chern class of a complex n-manifold is $c_1(\Lambda^{n,0}(M))$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

Calabi-Yau theorem

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a colomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\mathrm{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

Calabi-Yau theorem and Monge-Ampère equation

REMARK: Let (M,ω) be a Kähler n-fold, and Ω a non-degenerate section of $K(M)$, Then $|\Omega|^2 = \frac{\Omega\wedge\overline{\Omega}}{\omega^n}$ If ω_1 is a new Kaehler metric on (M,I), h, h_1 the associated metrics on $K(M)$, then $\frac{h}{h_1} = \frac{\omega_1^n}{\omega^n}$

COROLLARY: A metric $\omega_1 = \omega + \partial\overline{\partial}\varphi$ **is Ricci-flat if and only if** $(\omega + \partial\overline{\partial}\varphi)^n = \omega^n e^f$**,** where $-2\partial\overline{\partial} f = \Theta_{K,\omega}$.

Proof: For such f, φ, one has $\log\frac{h}{h_1} = -f$. This gives

$$\Theta_{K,\omega_1} = \Theta_{K,\omega} + \partial\overline{\partial}\frac{h}{h_1} = \Theta_{K,\omega} - 2\partial\overline{\partial} f = 0.$$

■

THEOREM: (Calabi-Yau) Let (M,ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that $(\omega + dd^c\varphi)^n = Ae^f\omega^n$, where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

REMARK:

$$(\omega + dd^c\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By dd^c-lemma, one has $\omega_2 = \omega_1 + dd^c\psi$. **We need to show** $\psi = const$**.**

Step 2: This gives

$$0 = (\omega_1 + dd^c\psi)^n - \omega_1^n = dd^c\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form** ω_3 **on** M **such that** $\omega_3^{n-1} = P$**.**

Step 4: Since $dd^c\psi \wedge P = 0$, this gives $\psi dd^c\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\overline{\partial}\psi \wedge P = -\int_M \partial\psi \wedge \overline{\partial}\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore** $\overline{\partial}\psi = 0$**.** ■