Kähler manifolds and holonomy

lecture 3

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Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Levi-Civita connection and Kähler geometry

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) (M, I, g) **is Kähler**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves** φ**.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point** $x \in M$**.**

The Berger's list

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

Chern connection

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: A section $b \in B$ is holomorphic iff $\overline{\partial}(b) = 0$

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}: B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider the Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$. We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Calabi-Yau manifolds

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a colomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\mathrm{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

REMARK: Converse is also true: **any Ricci-flat Kähler manifold has a finite covering which is Calabi-Yau.** This is due to Bogomolov.

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

REMARK: Its proof uses spinors (see below).

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$.**

DEFINITION: A holomorphically symplectic Ricci-flat Kaehler manifold is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Ricci-flat Riemannian manifold with $\pi_1(M)$ infinite. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

COROLLARY: A fundamental group of a compact Ricci-flat Riemannian manifold is **"virtually polycyclic": it is projected to a free abelian subgroup with finite kernel.**

REMARK: This is equivalent to any compact Ricci-flat manifold having a finite covering which has free abelian fundamental group.

REMARK: This statement contains the Bieberbach's solution of Hilbert's 18-th problem on classification of crystallographic groups.

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}ol(M_l) = Sp(n_l)$, $\mathcal{H}ol(K_l) = SU(m_l)$

Harmonic forms

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes ... \otimes x_k, x'_1 \otimes x'_2 \otimes ... \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)...g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g): $g(\alpha, \beta) := \int_M g(\alpha, \beta) \operatorname{Vol}_M$. The topology induced by this metric is called **L^2-topology.**

DEFINITION: Let d be the de Rham differential and d^* denote the adjoint operator. The **Laplace operator** is defined as $\Delta := dd^* + d^*d$. A form is called **harmonic** if it lies in $\ker \Delta$.

THEOREM: The image of Δ is closed in L^2-topology on differential forms.

REMARK: This is a very difficult theorem!

REMARK: On a compact manifold, the form η is **harmonic iff $d\eta = d^*\eta = 0$.** Indeed, $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$.

COROLLARY: This defines a map $\mathcal{H}^i(M) \xrightarrow{\tau} H^i(M)$ from harmonic forms to cohomology.

Hodge theory

THEOREM: (Hodge theory for Riemannian manifolds)
On a compact Riemannian manifold, the map $\mathcal{H}^i(M) \xrightarrow{\tau} H^i(M)$ to cohomology **is an isomorphism.**

Proof. Step 1: $\ker d \perp \operatorname{im} d^*$ and $\operatorname{im} d \perp \ker d^*$. Therefore, **a harmonic form is orthogonal to** $\operatorname{im} d$**.** This implies that τ **is injective**.

Step 2: $\eta \perp \operatorname{im} \Delta$ **if and only if** η **is harmonic.** Indeed, $(\eta, \Delta x) = (\Delta x, x)$.

Step 3: Since $\operatorname{im} \Delta$ is closed, **every closed form** η **is decomposed as** $\eta = \eta_h + \eta'$**,** where η_h is harmonic, and $\eta' = \Delta\alpha$.

Step 4: When η is closed, η' is also closed. Then $0 = (d\eta, d\alpha) = (\eta, d^* d\alpha) = (\Delta\alpha, d^* d\alpha) = (dd^*\alpha, d^* d\alpha) + (d^* d\alpha, d^* d\alpha)$. The term $(dd^*\alpha, d^* d\alpha)$ vanishes, because $d^2 = 0$, hence $(d^* d\alpha, d^* d\alpha) = 0$. This gives $d^* d\alpha = 0$, and $(d^* d\alpha, \alpha) = (d\alpha, d\alpha) = 0$. We have shown that **for any closed** η **decomposing as** $\eta = \eta_h + \eta'$**, with** $\eta' = \Delta\alpha$**,** α **is closed**

Step 5: This gives $\eta' = dd^*\alpha$, hence η **is a sum of an exact form and a harmonic form.** ■

REMARK: This gives a way of obtaining the Poincare duality via PDE.

Hodge decomposition on cohomology

THEOREM: *(this theorem will be proven in the next lecture)*
On a compact Kaehler manifold M, **the Hodge decomposition is compatible with the Laplace operator.** This gives a decomposition of cohomology, $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: $H^p(M)$ **is even-dimensional for odd** p**.**

The Hodge diamond:

$$
\begin{array}{ccccccc}
 & & & H^{n,n} & & & \\
 & & H^{n,n-1} & & H^{n-1,n} & & \\
 & H^{n,n-2} & & H^{n-1,n-1} & & H^{n-2,n} & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots \\
 & H^{2,0} & & H^{1,1} & & H^{0,2} & \\
 & & H^{1,0} & & H^{0,1} & & \\
 & & & H^{0,0} & & &
\end{array}
$$

REMARK: $H^{p,0}(M)$ **is the space of holomorphic** p**-forms.** Indeed, $dd^* + d^*d = 2(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})$ (next lecture), hence **a holomorphic form on a compact Kähler manifold is closed.**

Holomorphic Euler characteristic

DEFINITION: A holomorphic Euler characteristic $\chi(M)$ of a Kähler manifold is a sum $\sum(-1)^p \dim H^{p,0}(M)$.

THEOREM: (Riemann-Roch-Hirzebruch) For an n-fold, $\chi(M)$ **can be expressed as a polynomial expressions of the Chern classes,** $\chi(M) = td_n$ where td_n is an n-th component of the Todd polynomial,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

REMARK: The Chern classes are obtained as polynomial expression of the curvature (Gauss-Bonnet). Therefore $\chi(\tilde{M}) = p\chi(M)$ **for any unramified** p**-fold covering** $\tilde{M} \longrightarrow M$**.**

REMARK: Bochner's vanishing and the classical invariants theory imply:

1. When $\mathcal{H}ol(M) = SU(n)$, we have $\dim H^{p,0}(M) = 1$ **for** $p = 1, n$**, and 0 otherwise.** In this case, $\chi(M) = 2$ for even n and 0 for odd.

2. When $\mathcal{H}ol(M) = Sp(n)$,we have $\dim H^{p,0}(M) = 1$ **for even** p $0 \leqslant p \leqslant 2n$**, and 0 otherwise.** In this case, $\chi(M) = n + 1$.

COROLLARY: $\pi_1(M) = 0$ if $\mathcal{H}ol(M) = Sp(n)$, or $\mathcal{H}ol(M) = SU(2n)$. If $\mathcal{H}ol(M) = SU(2n+1)$, $\pi_1(M)$ is finite.

Spinors and Clifford algebras

DEFINITION: A Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = q(x, y)1$.

REMARK: A Clifford algebra of a complex vector space with $V = \mathbb{C}^n$ with q non-degenerate **is isomorphic to** $\operatorname{Mat}\left(\mathbb{C}^{n/2}\right)$ **(**n **even) and** $\operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right) \oplus \operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right)$ **(**n **odd).**

DEFINITION: The space of spinors of a complex vector space V, q is a fundamental representation of $Cl(V)$ (n even) and one of two fundamental representations of the components of $\operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right) \oplus \operatorname{Mat}\left(\mathbb{C}^{\frac{n-1}{2}}\right)$ (n odd).

REMARK: A 2-sheeted covering $\operatorname{Spin}(V) \longrightarrow SO(V)$ naturally acts on the spinor space, which is called **the spin representation of** $\operatorname{Spin}(V)$**.**

DEFINITION: Let Γ be a principal $SO(n)$-bundle of a Riemannian oriented manifold M. We say that M is a spin-manifold, **if Γ can be reduced to a** $Spin(n)$**-bundle.**

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(M)$ **vanishes.**

Spinor bundles and Dirac operator

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal $Spin(n)$-bundle and a spin representation.

DEFINITION: Consider the map $TM \otimes \text{Spin} \longrightarrow \text{Spin}$ induced by the Clifford multiplication. One defines **the Dirac operator** $D: \text{Spin} \longrightarrow \text{Spin}$ as a composition of $\nabla: \text{Spin} \longrightarrow \Lambda^1 M \otimes \text{Spin} = TM \otimes \text{Spin}$ and the multiplication.

DEFINITION: A **harmonic spinor** is a spinor ψ such that $D(\psi) = 0$.

THEOREM: (Bochner's vanishing) A harmonic spinor ψ on a compact manifold with vanishing scalar curvature $Sc = Tr(\text{Ric})$ **satisfies $\nabla\psi = 0$.**

Proof: The **coarse Laplacian** $\nabla^*\nabla$ is expressed through the Dirac operator using the **Lichnerowitz formula** $\nabla^*\nabla - D^2 = -\frac{1}{4}Sc$. When these two operators are equal, **any harmonic spinor ψ lies in $\ker \nabla^*\nabla$, giving $(\psi, \nabla^*\nabla\psi) = (\nabla\psi, \nabla\psi) = 0$.** ■

Bochner's vanishing on Kaehler manifolds

REMARK: **A Kaehler manifold is spin if and only if $c_1(M)$ is even,** or, equivalently, if there exists a square root of a canonical bundle $K^{1/2}$.

REMARK: On a Kaehler manifold of complex dimension n, **one has a natural isomorphism between the spinor bundle and $\Lambda^{*,0}(M)\otimes K^{1/2}$** (for n even) and $\Lambda^{2*,0}(M)\otimes K^{1/2}$ (for n odd).

REMARK: On a Kähler manifold, the Dirac operator corresponds to $\partial+\partial^*$.

COROLLARY: **On a Ricci-flat Kähler manifold, all $\alpha\in \ker(\partial+\partial^*)\Big|_{\Lambda^{*,0}(M)}$ ara parallel.**

REMARK: $\ker\partial+\partial^*=\ker\{\partial,\partial^*\}$, where $\{\cdot,\cdot\}$ denotes the anticommutator. However, $\{\partial,\partial^*\}=\{\overline{\partial},\overline{\partial}^*\}$ as Kähler identities imply. Therefore, **on a Kähler manifold, harmonic spinors are holomorphic forms**.

THEOREM: **(Bochner's vanishing)** Let M be a Ricci-flat Kaehler manifold, and $\Omega\in\Lambda^{p,0}(M)$ a holomorphic differential form. **Then $\nabla\Omega=0$.**