Minimal models for LCK manifolds

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Estruturas geométricas em variedades
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LCK manifolds

DEFINITION: A complex Hermitian manifold (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form**.

REMARK: This definition **is equivalent to the existence of a Kähler cover $(\tilde{M}, \tilde{\omega}) \to M$ such that the deck group Γ acts on $(M, \tilde{\omega})$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.** Let $\tilde{M}$ be a covering such that the pullback $\tilde{\theta}$ of θ is exact, $df = \tilde{\theta}$. Then the pullback of $\tilde{\omega}$ is conformal to a Kähler form $e^{-f}\tilde{\omega}$.

Vaisman theorem

REMARK: Let (M,ω,θ) be an LCK manifold, and θ' another 1-form, homologous to θ. Write $\theta' - \theta = df$. Then

$$d(e^f \omega) = e^f(d\omega + df \wedge \omega) = e^f(\theta \wedge \omega + df \wedge \omega) = \theta' \wedge (e^f \omega).$$

In other words, **conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1-form cohomologous to the Lee form is a Lee form of a conformally equivalent LCK metric.**

THEOREM: (Vaisman)
A compact LCK manifold (M,I,θ) with non-exact Lee form **does not admit a Kähler structure.**

Proof: On a compact manifold of Kähler type, any $[\theta] \in H^1(M,\mathbb{R})$ can be represented by α, obtained as a real part of a holomorphic form. This gives $d^c\alpha = 0$. After a conformal change of the metric, we can assume that $d\omega = \alpha \wedge \omega$, and $dd^c\omega = \alpha \wedge I(\alpha) \wedge \omega$. On a Kähler manifold, a positive exact form must vanish, which implies $\alpha \wedge I(\alpha) \wedge \omega = 0$ and $\alpha = 0$. ■

REMARK: Such manifolds are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Vaisman manifolds

DEFINITION: The LCK manifold (M, I, g, ω) is a **Vaisman manifold** if the Lee form is parallel with respect to the Levi-Civita connection.

THEOREM: A compact (strictly) LCK manifold M is Vaisman **if and only if it admits a non-trivial action of a complex Lie group of positive dimension,** acting by holomorphic isometries.

DEFINITION: **A linear Hopf manifold** is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

CLAIM: **All diagonal Hopf manifolds are Vaisman,** and **all non-diagonal Hopf manifolds are LCK and not Vaisman**.

EXAMPLE: **Almost all non-Kähler compact complex surfaces are LCK.** Among those, **only elliptic surfaces and some Hopf surfaces are Vaisman.**

THEOREM: A compact complex manifold admits a Vaisman structure **if and only if it admits a holomorphic embedding to a diagonal Hopf manifold.**

χ-automorphic functions

CLAIM: Conformally equivalent Kähler forms are proportional.

Proof: Let $e^f\omega$ and ω be Kähler forms. Then $0 = d(e^f\omega) = e^f\omega \wedge df$. A multiplication with ω defines an injective map $\Lambda^1(M) \xrightarrow{\wedge\omega} \Lambda^3(M)$, hence $e^f\omega \wedge df = 0$ implies $df = 0$. ■

COROLLARY: Let (M, ω, θ) be an LCK manifold, $(\tilde{M}, \tilde{\omega})$ its Kähler cover. **Then the deck transform group Γ acts on $\tilde{M}, \tilde{\omega}$ by homotheties.** ■

DEFINITION: Denote by $\chi : \Gamma \to \mathbb{R}^{>0}$ the corresponding character, $\gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega}$. A function φ on $\tilde{M}$ is called **χ-automorphic** if $\gamma^*\varphi = \chi(\gamma)\varphi$.

LCK manifolds with potential

DEFINITION: An LCK manifold is called **an LCK manifold with LCK potential** if the Kähler form $\tilde{\omega}$ on $\tilde{M}$ has a χ-automorphic potential, $\tilde{\omega} = dd^c\varphi$, where φ is a χ-automorphic function.

REMARK: A small deformation of an LCK manifold might be non-LCK. A small deformation of Vaisman might be non-Vaisman. **A small deformation of LCK with potential is LCK with potential.**

EXAMPLE: A linear Hopf manifold **admits an LCK structure with LCK potential** (Ornea-V.).

THEOREM: (Ornea-V.) A compact manifold M, $\dim_{\mathbb{C}} M > 2$ admits an LCK potential **if and only if M admits a holomorphic embedding to a Hopf manifold.**

REMARK: This property **can be used instead of the definition.**

REMARK: In dimension 2 this is also true, if we assume **the GSS conjecture** (generally assumed to be true).

Minimal models

DEFINITION: Let X, Y be compact complex varieties. A **meromorphic map** from X to Y is a closed complex subvariety $Z \subset X \times Y$ which projects to X and to Y surjectively, and bijectively to X in a dense, open subset. A **bimeromorphic map** is a closed complex subvariety $Z \subset X \times Y$ which projects to X and to Y surjectively, and bijectively in a dense, open subset.

DEFINITION: Let $M \to M_1$ be a proper, holomorphic, bimeromorphic map if irreducible complex varieties. In this case we say that M **is a resolution** of M_1.

DEFINITION: Let M be a compact complex variety. We say that M **is minimal** if any resolution map $M \to M_1$ is biholomorphic. A **minimal model** for M is a variety M_1 which is bimeromorphic to M and minimal. A **bimeromorphic model** of M is a compact complex variety which is bimeromorphic to M.

REMARK: Defining the minimal models, usually one asks for some restrictions on the singularities of M_1 (such as "terminal singularities"). Our definition is stronger, and **as such cannot be applied to (say) projective manifolds, or even complex surfaces.**

LCK manifolds with potential are minimal

THEOREM: (Ornea-V.) **An LCK manifold with potential is minimal.**
Proof: Slide 10, but first we establish some preliminary results.

THEOREM: A bimeromorphic map of complex manifolds induces an isomorphism of the fundamental groups.
Proof: J. Kollár, *Shafarevich maps and plurigenera of algebraic varieties,* Invent. Math. 113, (1993) 177-215, §7.8.1. ■

LEMMA: Let (M,ω) be a compact LCK manifold, $(\tilde M,\tilde\omega)$ its Kähler cover, and $Z\subset M$ a subvariety of positive dimension. Assume that the Kähler form $\tilde\omega$ is exact. **Then the image of $\pi_1(Z)$ in $\pi_1(M)$ contains an infinite cyclic subgroup.**

Proof. Step 1: Denote by $\tilde Z\subset\tilde M$ the cover of Z obtained by the homotopy lifting lemma. **If the image of $\pi_1(Z)$ in $\pi_1(M)$ is finite, the variety $\tilde Z$ is compact.** This is impossible, because $\tilde Z$ admits a Kähler form $\tilde\omega$ which is exact, bringing $0=\int_{\tilde Z}\tilde\omega^{\dim_{\mathbb C} Z}=\operatorname{Vol}(\tilde Z)>0$; a contradiction.

Step 2: By the same argument, the Kähler form $\tilde\omega$ restricted to $\tilde Z$ is not the pullback of a Kähler form on Z. **This implies that the deck transform group acts on $(\tilde Z,\tilde\omega\big|_{\tilde Z})$ by non-trivial homotheties,** implying that $\chi(\pi_1(Z))\subset\mathbb R^{>0}$ is non-trivial. ■

Normal varieties

DEFINITION: A complex variety X is called **normal** if any locally bounded meromorphic function on an open subset $U \subset X$ is holomorphic.

PROPOSITION 1: Let Z be a normal variety, and $\varphi : \ Z_1 \to Z$ a holomorphic, closed map such that $\varphi^{-1}(z)$ is finite for all z and bijective in a general point. **Then φ is bijective and φ^{-1} is holomorphic.**

Proof: Theorem 1.102, G.-M. Greuel, C. Lossen, E. I. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics, Springer, 2007. ■

Minimal models (2)

THEOREM: (Ornea-V.)
Let M, M_1 be compact complex manifolds and $\varphi: M_1 \dashrightarrow M$ a bimeromorphism. Assume that M is an LCK manifold, and let $(\tilde{M}, \tilde{\omega})$ its Kähler cover. Assume that the Kähler form $\tilde{\omega}$ on $\tilde{M}$ is exact. **Then φ is holomorphic.**

Proof. Step 1: Let $X \subset M \times M_1$ be the graph of φ. By definition, X is a complex subvariety of $M \times M_1$ which projects to M and M_1 bijectively in a general point. We denote by $\sigma: X \to M$, $\sigma_1: X \to M_1$ the projection maps. **To prove the theorem, we need to show that $\sigma_1^{-1}(z)$ is finite for all $z \in M_1$.** Then the theorem follows from Proposition 1, because M_1 is smooth, hence normal.

Step 2: Assume, on the contrary, that for some $z \in M_1$, its preimage $Z_1 := \sigma_1^{-1}(z)$ is positive-dimensional. Since the projection of $M \times M_1$ to M is bijective on the set $M \times \{z\}$, the set Z_1 projects to M holomorphically and bijectively. Let $Z \subset M$ be the image of Z_1 in M.

Step 3: As shown above, the image of $\pi_1(Z)$ in $\pi_1(M)$ contains an infinite order cyclic subgroup. Therefore, its image in $\pi_1(X) = \pi_1(M)$ also contains an infinite order cyclic subgroup. **This is impossible, because $\pi_1(X) = \pi_1(M) = \pi_1(M_1)$, and the projection of Z to M_1 is a point.** ■

Algebraic dimension

DEFINITION: Let M be a compact complex manifold, and $\mathfrak{Mer}(M)$ the field of meromorphic functions, globally defined on M. Transcendental dimension of $\mathfrak{Mer}(M)$ is called **the algebraic dimension of M**, denoted $a(M)$.

THEOREM: (Moishezon)
For any compact complex manifold M, one has $a(M) \leqslant \dim_{\mathbb{C}} M$. Moreover, **the equality $a(M) = \dim_{\mathbb{C}} M$ implies that M is bimeromorphic to a projective manifold.**

DEFINITION: A compact complex manifold M is called **Moishezon** if $a(M) = \dim_{\mathbb{C}} M$.

DEFINITION: Let M be a compact complex manifold. An **algebraic reduction map** is a dominant meromorphic map $M \dashrightarrow X$ with connected fibers such that X is Moishezon, and $\mathfrak{Mer}(M) = \mathfrak{Mer}(X)$.

THEOREM: The algebraic reduction map always exists. Moreover, it is holomorphic over a general point of X.

Proof: K. Ueno, LNM 439. ■

Isotrivial elliptic fibrations

DEFINITION: A dominant holomorphic map is called **an elliptic fibration** if its general fiber is an elliptic curve, and **isotrivial** if its general fibers are isomorphic elliptic curves.

THEOREM: (Ornea-Vuletescu-V.) Let M be an LCK complex manifold which satisfies $a(M) = \dim_{\mathbb{C}} M - 1$, and $M \dashrightarrow X$ its algebraic reduction. Consider a resolution M_1 of M such that the algebraic reduction $\pi : M_1 \to B$ is holomorphic. **Then M is an isotrivial elliptic fibration.** Moreover, the Lee form θ is non-exact on all fibers of π.

Proof. Step 1: Let $E_1 \subset M_1$ be the exceptional locus of the natural bimeromorphic map to M. By definition Π is almost holomorphic, hence $\Pi : E \to B$ is not dominant. Let $E_B \subset B$ be the closure of its image. For any curve $C \subset B$ which does not belong to E_B, consider the elliptic surface $S_C \subset M$ obtained as the closure of $\Pi^{-1}(C \backslash E_B)$. **The locally conformally Kähler form on M restricted to S_C can be globally conformally Kähler only if $\chi\big|_{\pi_1(S_C)}$ is trivial.**

Isotrivial elliptic fibrations (2)

Step 2: By Hironaka flattening theorem, we can always bimeromorphically replace $M \dashrightarrow X$ by a map $M' \dashrightarrow X$ which is flat, in particular, all its fibers have the same dimension. **Therefore, we may assume that $\pi: M_1 \to B$ is flat, hence equidimensional.**

Step 3: An LCK manifold cannot be covered by a family of globally conformally Kähler manifolds (L. Ornea, V. Vuletescu, V., *Blow-ups of locally conformally Kähler manifolds*, IMRN 2013, no. 12, 2809-2821, Lemma 3.1) hence the Lee form is non-exact on S_C. Vaisman theorem implies that S_C is non-Kähler. **A non-Kähler surface of algebraic dimension 1 is elliptic and isotrivial.** This implies that π is an isotrivial elliptic fibration. ■

Elliptic Vaisman manifolds

THEOREM: (Ornea-V.)
Let L be an ample bundle on a projective orbifold, such that the total space $\operatorname{Tot}^\circ(L)$ of non-zero vectors in L is smooth. Consider an automorphism $R_\alpha: \operatorname{Tot}^\circ(L) \to \operatorname{Tot}^\circ(L)$ multiplying each vector $v \in \operatorname{Tot}^\circ(L)$ by a complex number α, with $|\alpha| > 1$. **Then $\frac{\operatorname{Tot}^\circ(L)}{\langle R_\alpha \rangle}$ is an elliptic Vaisman manifold, and, moreover, all elliptic Vaisman manifolds can be obtained this way.**

THEOREM: (Ornea-Vuletescu-V.) Let M be an LCK complex manifold which satisfies $a(M) = \dim_{\mathbb{C}} M - 1$. **Then M is a resolution of an elliptic Vaisman manifold.**

Proof: No time for this now. ■