Balanced metrics on locally conformally Kahler manifolds

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Complex structures

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \to TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

The Hodge decomposition in linear algebra

DEFINITION: **The Hodge decomposition** $V\otimes_{\mathbb R}\mathbb C := V^{1,0}\oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb C} := V\otimes_{\mathbb R}\mathbb C$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb C} := \Lambda^n_{\mathbb R} V\otimes_{\mathbb R} C$ admits a decomposition

$$\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n} \Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0}\otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q}V$. The resulting decomposition $\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n}\Lambda^{p,q}V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t\cdot v + \sin t\cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

REMARK: The same construction **defines the Hodge decompostion on the de Rham algebra** of any almost complex manifold.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q} V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb C}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p} V$.

Further on, **TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i-forms.** It is a Grassman algebra on TM

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then $\partial := \frac{d+\sqrt{-1}\,d^c}{2}$, $\bar\partial := \frac{d-\sqrt{-1}\,d^c}{2}$ are the Hodge components of d**, $\partial = d^{1,0}$, $\bar\partial = d^{0,1}$.

Proof: Let V be a space generated by d, IdI. The natural action of $U(1)$ generated by $e^{\mathcal{W}}$ preserves V. **Since d has only two Hodge components. $U(1)$ acts with weights $\sqrt{-1}$ and $-\sqrt{-1}$**, and its Hodge components are expressed as above. ■

THEOREM: The following statements are equivalent.

1. I is integrable. 2. $\partial^2 = 0$. 3. $\bar\partial^2 = 0$. 4. $dd^c = -d^c d$ 5. $dd^c = 2\sqrt{-1}\,\partial\bar\partial$.

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: A closed complex submanifold of a Kähler manifold is Kähler.

REMARK: The Kähler condition is a way too strong, and **"majority" of compact complex manifolds are non-Kähler.**

Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n-manifold is called **Gauduchon** if $dd^c(\omega^{n-1}) = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique in any given conformal class, up to a constant multiplier.

REMARK: This is one of **very few statements** which is valid (and can be applied) to all compact complex manifolds.

REMARK: This is very useful, because allows to define **a degree** of a holomorphic bundle, define stability, and prove a **non-Kähler version of Donaldson-Uhlenbeck-Yau therem.**

Balanced and SKT metrics on complex manifolds

DEFINITION: For each $1 \leqslant k \leqslant n-1$, the condition $d(\omega^k) = 0$ implies $d\omega = 0$. Hermitian metric is called **balanced** if $d(\omega^{n-1}) = 0$. All twistor spaces are balanced (Hitchin). All Moishezon manifolds are balanced (Alessandrini-Bassaneli). The notion was introduced by Michelson in "On the existence of special metrics in complex geometry," Acta Math. 149 (1982).

DEFINITION: A metric g on a manifold M with $\dim_{\mathbb{C}} M > 2$is called **SKT ("strong Kähler torsion")** or **pluriclosed** if $dd^c\omega = 0$.

REMARK: SKT condition **is essential in the literature about generalized complex and generalized Kähler structures** (Hitchin, Gualtieri, Cavalcanti).

REMARK: In dimension 2 the condition $dd^c\omega = 0$ is the Gauduchon condition, and we always assume $\dim_{\mathbb{C}} M > 0$.

DEFINITION: A form ω is called **taming** or **symplectic-Hermitian** if it is a (1,1)-part of a symplectic form.

REMARK: Clearly, **a symplectic-Hermitian form is pluriclosed.** The converse is false. Indeed, **there are no examples of symplectic-Hermitian form on non-Kähler compact complex manifolds**; Streets-Tian conjectured they don't exist.

LCK manifolds

DEFINITION: A complex Hermitian manifold of dimension $\dim_{\mathbb{C}} > 1$ (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee formm** and its cohomology class **the Lee class**.

REMARK: This definition **is equivalent to the existence of a Kähler cover $(\tilde{M}, \tilde{\omega}) \to M$ such that the deck group Γ acts on $(M, \tilde{\omega})$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.** Let $\tilde{M}$ be a covering such that the pullback $\tilde{\theta}$ of θ is exact, $df = \tilde{\theta}$. Then the pullback of $\tilde{\omega}$ is conformal to a Kähler form $e^{-f}\tilde{\omega}$.

REMARK: All known compact LCK manifolds belong to one of three classes: blow-ups of LCK with potential, blow-ups of Oeljeklaus-Toma and Kato. I will define these three classes later in this talk.

The main result today:

THEOREM: Let (M, I) be a compact complex non-Kähler LCK-manifold which is birational to either LCK with potential, Oeljeklaus-Toma or Kato manifold. **Then (M, I) does not admit a balanced metric.**

SKT, balanced, LCK properties are exclusive

CONJECTURE: Let M be a compact complex manifold which admits Hermitian forms ω_1 and ω_2 which belong to two classes in the set $\{$SKT, balanced, LCK$\}$. **Then M admits a Kähler structure.**

A weaker form of this statement is not hard to prove.

THEOREM: Let (M, I, ω) be a compact complex Hermitian n-manifold. Assume that ω is either

(a). SKT and LCK,
(b). balanced and LCK,
(c). SKT and balanced.

Then ω is Kähler.

SKT, balanced, LCK properties are exclusive

THEOREM: Let (M, I, ω) be a compact complex Hermitian n-manifold. Assume that ω is either
(a). SKT and LCK,
(b). balanced and LCK,
(c). SKT and balanced.
Then ω is Kähler.

Proof of (a): Assume (M, I, ω) is SKT and LCK. Let $\theta \in \Lambda^1(M)$ be the Lee form, $d\omega = \omega \wedge \theta$. Let $\theta^c := I(\theta)$. Then $d^c\omega = I^{-1}dI(\omega) = I^{-1}(\theta\wedge\omega) = -\theta^c\wedge\omega$. This gives

$$0 = d^c d\omega = d^c(\theta\omega) = d(d^c\theta) \wedge \omega - \theta \wedge d^c\omega = dd^c(\theta) \wedge \omega + \theta \wedge \theta^c\omega \quad (*)$$

Since $\dim_{\mathbb{C}} M > 2$, the multiplication map $\eta \mapsto \eta \wedge \omega$ is injective, hence (*) implies that $dd^c(\theta) = -\theta \wedge \theta^c$. Then

$$dd^c\omega^{n-1} = (n-1)dd^c(\theta)\wedge\omega^{n-1} - (n-1)^2\theta\wedge\theta^c\wedge\omega^{n-1} = -(n-1)(n-2)\theta\wedge\theta^c\wedge\omega^{n-1}.$$

However, $\theta \wedge \theta^c \wedge \omega^{n-1} = 2n|\theta|^2\omega^n$. This brings a contradiction:

$$0 = \int_M dd^c\omega^{n-1} = \int_M -(n-1)(n-2)\theta\wedge\theta^c\wedge\omega^{n-1} = -\frac{(n-1)(n-2)}{2}n\int_M |\theta|\wedge\omega^n$$

The last integral vanishes if and only if $\theta = 0$, hence ω is closed.

SKT, balanced, LCK properties are exclusive (2)

Proof of (b): Assume (M, I, ω) is balanced and LCK. Then $0 = d\omega^{n-1} = (n-1)\theta \wedge \omega^{n-1}$. However, the multiplication map $\eta \mapsto \eta \wedge \omega^{n-1}$ is an isomorphism for all η and any Hermitian ω (do this as an exercise), hence again $\theta = 0$.

Proof of (c). Step 1: Assume (M, I, ω) is balanced and SKT. Then $d(\omega^{n-1}) = (d\omega) \wedge \omega^{n-2} = 0$, and $dd^c\omega = 0$, hence $d\omega$ and $d^c\omega$ are d and d^c-closed. The equation $(d\omega) \wedge \omega^{n-2} = 0$ implies that $d\omega$ is **primitive**, that is, satisfies $\Lambda_\omega(d\omega) = 0$, where $\Lambda_\omega = L_\omega^*$, and $L_\omega(\eta) := \omega \wedge \eta$. This form is of Hodge type $(1,2)+(2,1)$ because ω is of type (1,1), and de Rham differential shifts the Hodge grading at most by 1.

Step 2: By Hodge-Riemann relations, any primitive $(1,2)+(2,1)$ real form α satisfies $\alpha \wedge I(\alpha) \wedge \omega^{n-3} = -C|\alpha|^2\omega^n$, where C is a positive rational constant.

Step 3: Let $\alpha := d\omega$. Since ω is SKT, we have

$$0 = dd^c(\omega^{n-1}) = (n-1)(n-2)d\omega \wedge d^c\omega \wedge \omega^{n-3} = -(n-1)(n-2)C|\alpha|^2\omega^n \quad (**)$$

(by step 1, α is a primitive $(1,2)+(2,1)$-form, then (**) follows from Step 2), hence $d\omega = 0$. ■

Vaisman theorem

REMARK: Let (M,ω,θ) be an LCK manifold, and θ' another 1-form, homologous to θ. Write $\theta' - \theta = df$. Then

$$d(e^f\omega) = e^f(d\omega + df\wedge\omega) = e^f(\theta\wedge\omega + df\wedge\omega) = \theta'\wedge(e^f\omega).$$

In other words, **conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1-form cohomologous to the Lee form is a Lee form of a conformally equivalent LCK metric.**

THEOREM: (Vaisman)
A compact LCK manifold (M,I,θ) with non-exact Lee form **does not admit a Kähler structure.**

Proof: On a compact manifold of Kähler type, any $[\theta]\in H^1(M,\mathbb{R})$ can be represented by α, obtained as a real part of a holomorphic form. This gives $d^c\alpha = 0$. After a conformal change of the metric, we can assume that $d\omega = \alpha\wedge\omega$, and $dd^c\omega = \alpha\wedge I(\alpha)\wedge\omega$. **On a Kähler manifold, a positive exact form must vanish, which implies $\alpha\wedge I(\alpha)\wedge\omega = 0$ and $\alpha = 0$.** ■

REMARK: Such manifolds are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Izu Vaisman

Izu Vaisman, b. June 22, 1938 in Jassy, Romania

Vaisman manifolds

DEFINITION: An LCK manifold is a **Vaisman manifold** if it admits a continuous action of complex isometries.

REMARK: This is actually a theorem, due to many autors, primarily Kamishima, Ornea, Istrati, V.; the original definition is that "(M, I, g, ω) is Vaisman if the Lee form θ is parallel with respect to the Levi-Civita connection."

EXAMPLE: **All non-Kähler elliptic surfaces are Vaisman.**

DEFINITION: **A linear Hopf manifold** is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

EXAMPLE: **All diagonal Hopf manifolds are Vaisman,** and **when A cannot be diagonalized, M is LCK and not Vaisman**.

THEOREM: **(Ornea-V.)**
All complex submanifolds of Vaisman manifolds are Vaisman. All Vaisman manifolds admit a holomorphic embedding to a diagonal Hopf manifold (which is Vaisman, too).

LCK manifolds with potential

DEFINITION: An LCK manifold is called **an LCK manifold with LCK potential** if the Kähler form $\tilde\omega$ on $\tilde M$ has a χ-automorphic potential, $\tilde\omega = dd^c\varphi$, where φ is a χ-automorphic function.

REMARK: A small deformation of an LCK manifold might be non-LCK. A small deformation of Vaisman might be non-Vaisman. **A small deformation of LCK with potential is LCK with potential.**

EXAMPLE: All Hopf manifolds **admit an LCK structure with LCK potential** (Ornea-V.).

THEOREM: (Ornea-V.) A compact manifold M, $\dim_{\mathbb C} M > 2$ admits an LCK potential **if and only if M admits a holomorphic embedding to a Hopf manifold.**

REMARK: This property **can be used instead of the definition.**

REMARK: In dimension 2 **this is also true if we assume the GSS conjecture**.

Normed fields

DEFINITION: An absolute value on a field k is a function $|\cdot|: k \to \mathbb{R}^{\geqslant 0}$, satisfying the following

1. Zero: $|x| = 0 \Leftrightarrow x = 0$.

2. Multiplicativity: $|xy| = |x||y|$.

3. There exists $c > 0$ such that $|\cdot|^c$ satisfies the triangle inequality.

EXAMPLE: The usual absolute value on $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

EXAMPLE: Let p – be a prime number, and $m, n \in \mathbb{Z}$ coprime with p. Define **p-adic absolute value** on $\mathbb{Q}$ via $|\frac{m}{n}p^k| := p^{-k}$.

REMARK: p-adic absolute value satisfies an additional "non-archimedean axiom": $|x+y| \leqslant \max(|x|, |y|)$. Such absolute values are called **non-archimedean**.

REMARK: Any power of non-archimedean absolute value is again non-archimedean, and satisfies the triangle inequality.

Normed fields and topology

DEFINITION: Let $|\cdot|$ be an absolute value on a field F. Consider topology on F with open sets generated by

$$B_\varepsilon(x) := \{y \in k \ \ | \ \ |x - y| < \varepsilon\}.$$

Absolute values are called **equivalent** if they induce the same topology.

THEOREM: Absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if and only if $|\cdot|_1 = |\cdot|_2^c$ for some $c > 0$.

THEOREM: (Ostrowski) **Every absolute value on $\mathbb{Q}$ is equivalent to the usual ("archimedean") one or to p-adic one.**

DEFINITION: A completion of a field k under an absolute value $|\cdot|$ is a completion of k in a metric $|\cdot|^c$, where $c > 0$ is a constant such that $|\cdot|^c$ satisfies the triangle inequality.

REMARK: A completion of a field is again a field.

EXAMPLE: A completion of $\mathbb{Q}$ under the p-adic absolute value is called **a field of p-adic numbers**, denoted $\mathbb{Q}_p$.

Local fields

DEFINITION: A finite extension $K:k$ of fields is a field $K \supset k$ which is finite-dimensional as a vector space over k. **A number field** is a finite extension of $\mathbb{Q}$. **Functional field** is a finite extension of $\mathbb{F}_p(t)$. **Global field** is a number or functional field. **Local field** is a completion of a global field under a non-trivial absolute value.

THEOREM: Let $\overline{k}$ be a field which is complete and locally compact under some absolute value. **Then $\overline{k}$ is a local field**.

DEFINITION: Let $K{:}k$ be a finite extension, and $x \in K$. Consider the multiplication by x as a k-linear endomorphism of K. Define **the norm** $N_{K/k}(x)$ as a determinant of this operator.

REMARK: Norm defines a homomorphism of multiplicative groups $K^* {\rightarrow} k^*$.

REMARK: For Galois extensions, the norm $N_{K/k}(x)$ **is a product of all elements conjugate to x.**

THEOREM: Let $\overline{K}:\overline{k}$ be a finite extension of local fields, degree n. **Then an absolute value on $\overline{k}$ is uniquely extended to $\overline{K}$.** Moreover, **this extension is expressed as $|x| := \left|N_{K/k}(x)\right|^{\frac{1}{n}}$.**

Absolute values and extensions of global fields

CLAIM: Let A, B be extensions of a field k, of characteristic 0 where $A{:}k$ is finite. Consider $A \otimes_k B$ as an k-algebra. **Then $A \otimes_k B$ is a direct sum of fields, containing A and B.**

THEOREM: Let k be a number field, $|\cdot|$ an absolute value, $K{:}k$ a finite extension, and $\overline{k}$ – its completion. Consider a decomposition $K \otimes_k \overline{k}$ into a direct sum of fields $K \otimes_k \overline{k} := \bigoplus_i \overline{K}_i$. **Then each extension of an absolute value $|\cdot|$ from k to K is induced from some $\overline{K}_i$, and all such extensions are non-equivalent.**

REMARK: When $k = \mathbb{Q}$, and $|\cdot|$ is the usual (archimedean) absolute value, we obtain that all K_i are extensions of $\mathbb{R}$, that is, isomorphic to $\mathbb{R}$ or $\mathbb{C}$. This gives

COROLLARY: **For each number field K of degree n over $\mathbb{Q}$, there exists only a finite number of different homomorphisms $K \hookrightarrow \mathbb{C}$**, all of them injective. Denote by s the number of embeddings whose image lies in $\mathbb{R} \subset \mathbb{C}$ (such an embedding is called **real**), and $2t$ the number of embedding, whose image does not lie in $\mathbb{R}$ ("**complex embeddings**"). Then $s + 2t = n$.

Dirichlet unit theorem

DEFINITION: Let $K:\mathbb{Q}$ be a number field of degree n. **The ring of integers** $\mathcal{O}_K \subset K$ is an integral closure of $\mathbb{Z}$ in K, that is, the set of all roots in K of monic polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + ... + a_0$ with integer coefficients $a_i \in \mathbb{Z}$.

CLAIM: An additive group $\mathcal{O}_K^+$ is a finitely generated abelian group of rank n.

DEFINITION: A unit of a ring $\mathcal{O}_K$ is an element $u \in \mathcal{O}_K$, such that u^{-1} also belongs to $\mathcal{O}_K$.

REMARK: $u \in \mathcal{O}_K$ is a unit if and only if the norm $N_{K/\mathbb{Q}}(x) \in \mathbb{Z}$ is invertible, that is, $N_{K/\mathbb{Q}}(x) = \pm 1$.

Dirichlet's unit theorem: Let K be a number field which has s real embeddings and $2t$ complex ones. Then **the group of units $\mathcal{O}_K^*$ is isomorphic to $G \times \mathbb{Z}^{t+s-1}$,** where G is a finite group of roots of unity contained in K. Moreover, if $s > 0$, one has $G = \pm 1$.

REMARK: For a quadratic field, the group of units is a group of solutions of Pell's equation.

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embedding denoted $\tau_i, \overline{\tau}_i$ and s real ones denoted σ_i, $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+}$ a free abelian subgroup $\mathcal{O}_K^{*,U}$ of rank s such that the quotient $\mathbb{R}^s/\mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to $\mathbb{R}^t$ as $\xi \to \Big(\log(\sigma_1(\xi)), ..., \log(\sigma_t(\xi))\Big)$. Let $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: An Oeljeklaus-Toma manifold is a quotient $\mathbb{C}^t \times H^s/\Gamma$, where $\mathcal{O}_K^+$ acts on $\mathbb{C}^t \times H^t$ as

$$\zeta(x_1, ..., x_t, y_1, ..., y_s) = \Big(x_1 + \tau_1(\zeta), ..., x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), ..., y_s + \sigma_s(\zeta)\Big),$$

and $\mathcal{O}_K^{*,U}$ as

$$\xi(x_1, ..., x_t, y_1, ..., y_s) = \Big(x_1, ..., x_t, \sigma_1(\xi)y_1, ..., \sigma_t(\xi)y_t\Big)$$

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times H^s/\Gamma$ **is a compact complex manifold,** without any non-constant meromorphic functions. When $t = 1$, it is locally conformally Kähler.

Kato manifolds

DEFINITION: Let B be an open ball in $\mathbb{C}^n$, $n > 1$, and $\tilde{B} \overset{\pi}{B}$ be a bimeromorphic, holomorphic map, which is an isomorphism outside of a compact subset. Remove a small ball in $\tilde{B}$ and glue it to the boundary of $\tilde{B}$, extending the complex structure smoothly (and holomorphically) on the resulting manifold, denoted by M. Then M us called **a Kato manifold.**

THEOREM: (Brunella) Suppose that M is a Kato manifold obtained from $\tilde{B} \overset{\pi}{B}$ with $\tilde{B}$ Kähler. **Then M is LCK.**

THEOREM: (Kato) Let M be a Kato manifold. Then there exists a family M_t of complex manifolds over a punctured disk **such that $M = M_0$ and all other M_t are bimeromorphic to a Hopf manifold.**

DEFINITION: Let M be a complex manifold, and $\Gamma \subset M$ be an open subset which is isomorphic as a complex manifold to a small neighbourhood of a sphere $S^{2n-1} \subset \mathbb{C}^n$. The set Γ is called **a global spherical shell** if the complement $M \backslash \Gamma$ is connected.

THEOREM: (Kato) Let M be a compact complex manifold. Then **M is a Kato manifold if and only if it contains a global spherical shell.**

Bott-Chern cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}M)$. Then $H^{p,q}_{BC}(M)$ is called **the Bott–Chern cohomology** of M.

REMARK: A (p,q)-form η is closed if and only if $\partial\eta = \overline{\partial}\eta = 0$. Using $2\sqrt{-1}\,\partial\overline{\partial} = dd^c$, **we could define the Bott–Chern cohomology $H^*_{BC}(M)$ as $H^*_{BC}(M) := \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}$.**

REMARK: There are natural (and functorial) maps from the Bott–Chern cohomology to the Dolbeault cohomology $H^*(\Lambda^{*,*}M, \overline{\partial})$ and to the de Rham cohomology, **but no morphisms between the de Rham and the Dolbeault cohomology.**

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

This result can be deduced from

THEOREM: There is an exact sequence,

$$H^*(\Lambda^{p,q-1}(M), \overline{\partial}) \oplus \overline{H^*(\Lambda^{q,p-1}(M), \overline{\partial})} \to H^{p,q}_{BC}(M) \to H^{p+q}(M).$$

with the second arrow mapping a class represented by a closed form to a class represented by the same form, and the first taking $(x,y) \in H^*(\Lambda^{p,q-1}(M), \overline{\partial}) \oplus \overline{H^*(\Lambda^{q,p-1}(M), \overline{\partial})}$ to $\partial x + \overline{\partial} y$.

Degree of an element in $H^{1,1}_{BC}(M)$ and Gauduchon metrics

REMARK: If ω is Gauduchon, then (by Stokes' theorem) $\int_M \omega^{n-1} dd^c f = 0$ for any function f with compact support. Therefore, $\int_M \omega^{n-1} \wedge \alpha$ is a functional on $H^{1,1}_{BC}(M)$. This functional is called **the degree**.

EXAMPLE: Let Θ_L be the curvature of Chern connection on a holomorphic line bundle L. Since $\Theta_L = -dd^c|l|$, where l is a holomorphic section of L, the curvature is well defined up to to $dd^c \log|h|$, where h is a conformal factor given by a ration of two Hermitian metrics. Therefore, **for any line bundle L, the quantity $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$ is well defined.**

REMARK: This is **the starting point of the Kobayashi-Hitchin correspondence on complex manifold.**

Degree of an element in $H^1(M,\mathbb{R})$ and Gauduchon metrics

DEFINITION: Let M be a compact complex manifold, and ω a Gauduchon form. Consider the natural map $H^1(M,\mathbb{R}) \to H^{1,1}_{BC}(M)$ which takes a closed real 1-form α to the Bott-Chern class of $d^c\alpha$. Locally, α is df, hence $d^c\alpha$ is a (1,1)-form. **Define** $\deg\alpha$ **as** $-\int_M \omega^{n-1}\wedge d^c\alpha$**.** This is a well defined functional on first cohomology.

CLAIM: In these assumptions, **the form $d^c\alpha \in \Lambda^2(M)$ is always exact.**

Proof: Indeed, $d^c\alpha = IdI\alpha = d(I\alpha)$ because $I(\beta) = \beta$ for any (1,1)-form β. ■

DEFINITION: Let M be a compact complex manifold admitting an LCK structure. Define its **Lee cone** as the set of all classes $[\theta] \in H^1(M,\mathbb{R})$ of all the Lee forms for all LCK structures.

THEOREM: Let M be a compact LCK manifold with potential, and u a Gauduchon form. **Then its Lee cone is the set of all $\alpha \in H^1(M)$ such that $\deg_u \alpha > 0$, where $\deg_u$ is the degree map associated with u.**

Aeppli cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{AE}(M)$ the space of dd^c-closed (p,q)-forms modulo $\partial(\Lambda^{p-1,q}M)+\overline{\partial}(\Lambda^{p,q-1}M)$. Then $H^{p,q}_{AE}(M)$ is called **the Aeppli cohomology** of M.

THEOREM: Let M be a compact complex n-manifold. Then **the Aeppli cohomology is finite-dimensional.** Moreover the natural pairing $H^{p,q}_{BC}(M)\times H^{n-p,n-q}_{AE}(M){\to}H^{2n}(M)=\mathbb{C}$, taking x,y to $\int_M x\wedge y$ **is non-degenerate and identifies $H^{p,q}_{BC}(M)$ with the dual $H^{n-p,n-q}_{AE}(M)^*$.**

The Gauduchon cone

DEFINITION: Let M be a complex manifold, and ω a Gauduchon metric. **A Gauduchon form** of M is ω^{n-1}.

CLAIM: Fix a positive volume form Vol on M. A form $\eta\in \Lambda^{n-1,n-1}(M,\mathbb{R})$ defines a Hermitian form on $\Lambda^1(M)$ taking x,y to $\frac{\eta\wedge x\wedge y}{\mathrm{Vol}}$. **Then this Hermition form is positive definite if and only if $\eta=\alpha^{n-1}$,** where α is a Hermitian form.

REMARK: This result **implies that the set of all Gauduchon forms is a convex cone in $\Lambda^{n-1,n-1}(M,\mathbb{R})$.**

DEFINITION: The Gauduchon cone of a compact complex n-manifold is the set of all classes $\omega^{n-1}\in H^{n-1,n-1}_{AE}(M)$ of all Gauduchon forms.

DEFINITION: Recall that **pseudoeffective cone** $P\subset H^{1,1}_{BC}(M)$ is the cone of all Bott-Chern classes of all positive, closed (1,1)-currents.

THEOREM: (Lamari)
The Gauduchon cone is dual to the pseudoeffective cone.

Lee cone and the Gauduchon cone

Conjecture 2: Let θ be a Lee class on a compact LCK manifold, and u a Gauduchon metric. **Then** $\deg_u \theta > 0$**.** In other words, $d^c\theta$ is pseudo-effective.

THEOREM: Let (M, ω, θ) be a compact complex non-Kähler LCK-manifold which is birational to either LCK with potential, Oeljeklaus-Toma or Kato manifold, and u a Gauduchon metric. **Then** $\deg_u \theta \geqslant 0$**.**

Proof. Step 1: Theorem is true for free for OT and Vaisman manifolds, because $-d^c\theta$ is a positive, exact (1,1)-form. It is also true for all manifolds which are bimeromorphic to Vaisman and OT, because these manifold are blow-ups of Vaisman and OT, (Ornea-V.), and the pullback of a positive form is positive. **This implies that** $\deg_u \theta > 0$ in all these situations.

Lee cone and the Gauduchon cone (2)

THEOREM: Let (M,ω,θ) be a compact complex non-Kähler LCK-manifold which is birational to either LCK with potential, Oeljeklaus-Toma or Kato manifold, and u a Gauduchon metric. **Then $\deg_u \theta \geqslant 0$.** In other words, $d^c\theta$ is pseudo-effective.

Step 2: Suppose we have a smooth family (M_t,ω_t,θ_t) of LCK manifolds such that the statement of the theorem is true for all $t\neq 0$. Fix a Gauduchon metric u_0 on the central fiber M_0. We can extend it u_0 to a smooth family of Gauduchon metrics using the Gauduchon theorem. **On all fibers except the central, we have $\deg_{u_t}\theta_t \geqslant 0$, hence the same is true on the central fiber.**

Step 3: An LCK manifold with potential has a deformation (M_t,ω_t,θ_t) with all fibers except the central one Vaisman (Ornea-V.). The Kato manifold has a deformation (M_t,ω_t,θ_t) with all the fibers except the central one a blown-up Hopf (Kato). However, blown-up Hopf is bimeromorphic to LCK with potential, hence it also satisfies the statement of the theorem. ■

Proof of main theorem

THEOREM: Let (M, ω, θ) be a compact complex non-Kähler LCK-manifold which is birational to either LCK with potential, Oeljeklaus-Toma or Kato manifold. **Then (M, ω, θ) does not admit a balanced metric.**

Proof: Let α be a positive current which is Bott-Chern cohomologous to $-d^c\theta_0$, and ρ the balanced metric. Then $\int_M -d^c\theta_0 \wedge \rho^{n-1} = \int_M \alpha \wedge \rho^{n-1} = 0$, because $d^c\theta_0$ is exact. This implies that $\alpha = 0$: the mass of a non-zero positive current is always strictly positive. Therefore, $d^c\theta$ is Bott-Chern exact, impying that $d^c\theta = dd^c f$, and θ is cohomologous to a d, d^c-closed form θ_1. The same argument as in the proof of Vaisman theorem immediately implies that $\theta_1 = 0$ and M is Kähler. ■