Do products of compact complex manifolds admit LCK metrics?

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Estruturas geométricas em variedades
April 6, 2023, IMPA

joint work with Liviu Ornea and Victor Vuletescu

LCK manifolds

DEFINITION: A complex Hermitian manifold of dimension $\dim_{\mathbb{C}} > 1$ (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form**.

REMARK: This definition **is equivalent to the existence of a Kähler cover $(\tilde{M}, \tilde{\omega}) \to M$ such that the deck group Γ acts on $(M, \tilde{\omega})$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.** Let $\tilde{M}$ be a covering such that the pullback $\tilde{\theta}$ of θ is exact, $df = \tilde{\theta}$. Then the pullback of $\tilde{\omega}$ is conformal to a Kähler form $e^{-f}\tilde{\omega}$.

Vaisman theorem

REMARK: Let (M, ω, θ) be an LCK manifold, and θ' another 1-form, homologous to θ. Write $\theta' - \theta = df$. Then

$$d(e^f \omega) = e^f(d\omega + df \wedge \omega) = e^f(\theta \wedge \omega + df \wedge \omega) = \theta' \wedge (e^f \omega).$$

In other words, **conformally equivalent LCK metric give rise to homologous Lee forms, and any closed 1-form cohomologous to the Lee form is a Lee form of a conformally equivalent LCK metric.**

THEOREM: (Vaisman)
A compact LCK manifold (M, I, θ) with non-exact Lee form **does not admit a Kähler structure.**

Proof: On a compact manifold of Kähler type, any $[\theta] \in H^1(M, \mathbb{R})$ can be represented by α, obtained as a real part of a holomorphic form. This gives $d^c\alpha = 0$. After a conformal change of the metric, we can assume that $d\omega = \alpha \wedge \omega$, and $dd^c\omega = \alpha \wedge I(\alpha) \wedge \omega$. **On a Kähler manifold, a positive exact form must vanish, which implies $\alpha \wedge I(\alpha) \wedge \omega = 0$ and $\alpha = 0$.** ■

REMARK: Such manifolds are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Izu Vaisman

Izu Vaisman, b. June 22, 1938 in Jassy, Romania

LCK structures on a product

Further on, **when I say "a manifold is a product of two manifolds", I always assume "of positive dimension".**

Clearly, **any submanifold of an LCK manifold is LCK**. This implies that any LCK manifold which is a product of two complex manifolds is a product of LCK manifolds (a priori non-strict).

THEOREM: (Ornea-V.-Vuletescu)
Let M be an LCK manifold which is biholomorphic to a product of two complex manifolds, $M = X \times Y$. **Then both X and Y are strict LCK**.

The main question we discuss today: **Can a product of two complex manifolds of positive dimension admit an LCK structure?**

LCK structures on a product (2)

Can a product of two complex manifolds of positive dimension admit an LCK structure?

The answer is "most likely, not", but it is still not proven. However, up to a fundamental statement known as "GSS conjecture" (still not proven, but generally assumed to be true), we can prove this statement when one of the summands has dimension $\leqslant 2$.

GSS conjecture (more about it later today) claims that all complex surfaces with $b_1 = 1$ fall into one of three known classes, constructed explicitly. **When $b_1 = 3$ or more, it is automatically elliptic, and therefore LCK.**

THEOREM: (Ornea-V.-Vuletescu)
Let $M = X \times Y$, where M is an LCK manifold, and X a surface of one of the known classes (any surface if GSS conjecture is true). **Then M is Kähler.**

Non-Kähler surfaces

REMARK: A **complex surface** is a compact complex manifold of complex dimension 2.

THEOREM: (follows from Kodaira classification; a direct proof is due to Buchdahl and Lamari) A complex surface M admits a Kähler structure if and only if $b_1(M)$ is even.

DEFINITION: A complex surface M is called **class VII** if $b_1(M) = 1$. It is called **elliptic** if it admits a holomorphic map to a compact curve with fiber an elliptic curve.

THEOREM: (Belgun, Ornea-V.-Vuletescu) A complex, non-Kähler surface with $b_1(M) > 1$ is elliptic and LCK.

The GSS conjecture

DEFINITION: Let $S \subset \mathbb{C}^2$ be a standard sphere, and S_ε its ε-neighbourhood. A complex surface M **admits a global spherical shell** if there is a holomorphic embedding $S_\varepsilon \to M$, for some $\varepsilon > 0$, such that the complement of its image is connected. A surface admitting a global spherical shell is called **a Kato surface**.

REMARK: Kato surfaces **can be constructed explicitly from germs of birational automorphisms of $\mathbb{C}^2$.**

CONJECTURE: (GSS conjecture, due to Kato)
Let M be a class VII surface with $b_2 > 0$. **Then M is a Kato surface.**

REMARK: When $b_2 = 0$, the structure theorem for surfaces of class VII is due to Bogomolov (later today).

REMARK: By results of G. Dlousky, K. Oeljeklaus and M. Toma, a Kato surface M admits at least $b_2(M)$ distinct rational curves, and, conversely, **if a complex surface admits $b_2(M)$ distinct rational curves in a certain configuration, it is a Kato surface.**

THEOREM: (Brunella)
Kato surfaces admit an LCK structure.

Inoue surfaces

DEFINITION: Consider the action of $\mathbb{Z}$ on $\mathbb{Z}^3$ associated with a linear operator $A \in SL(3, \mathbb{Z})$ with one of the real eigenvalues > 1, and let Γ_{S_0} be the corresponding semidirect product $\Gamma_{S_0} := \mathbb{Z}^3 \rtimes \mathbb{Z}$. Let Heis be the group of upper triangular integer matrices 3×3, $Z(\text{Heis}) = \mathbb{Z}$ its center, and let $\mathbb{Z}$ act on Heis by automorphisms with real eigenvalues $\alpha, \alpha^{-1} \neq \pm 1$ on $\mathbb{Z}^2 = \text{Heis}/Z(\text{Heis})$. We denote by $\Gamma_{S_+} := \text{Heis} \rtimes \mathbb{Z}$ the corresponding semidirect product when $\alpha > 0$ and $\Gamma_{S_-} := \text{Heis} \rtimes \mathbb{Z}$ when $\alpha < 0$.

DEFINITION: Let $\mathbb{H}$ be the upper half-plane. Consider the groups Γ_{S_0}, $\Gamma_{S_\pm}$ acting cocompactly, faithfully and discontinuously on $\mathbb{C} \times \mathbb{H}$. The quotient manifolds $\mathbb{C} \times \mathbb{H}/\Gamma_{S_0}$, $\mathbb{C} \times \mathbb{H}/\Gamma_{S_+}$, $\mathbb{C} \times \mathbb{H}/\Gamma_{S_-}$ are called **the Inoue surfaces of class S_0, S_+, S_-.**

Bogomolov's theorem about surfaces of class VII

DEFINITION: a Hopf surface is a quotient of $\mathbb{C}^2\backslash 0$ by an action of $\mathbb{Z}$ which acts on $\mathbb{C}^2$ by holomorphic contractions.

THEOREM: (F. Belgun)
Inoue surfaces of class S_0 and S_- are LCK. Inoue surfaces of class S_+ are LCK if and only if they are double covers of S_-.

DEFINITION: a Hopf surface is a quotient of $\mathbb{C}\times\mathbb{H}$ by an action of $\mathbb{Z}$ which acts on $\mathbb{C}^2$ by holomorphic contractions.

THEOREM: (Bogomolov, Li-Yau-Zheng, Teleman)
Let M be a class VII surface with $b_2 = 0$. **Then M is a Hopf surface or an Inoue surface.**

Vaisman manifolds

DEFINITION: An LCK manifold is a **Vaisman manifold** if it admits a continuous action of complex isometries.

REMARK: This is actually a theorem, due to many autors, primarily Kamishima, Ornea, Istrati, V.; the original definition is that "(M, I, g, ω) is Vaisman if the Lee form θ is parallel with respect to the Levi-Civita connection."

EXAMPLE: All non-Kähler elliptic surfaces are Vaisman.

DEFINITION: A linear Hopf manifold is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

EXAMPLE: All diagonal Hopf manifolds are Vaisman, and **all non-diagonal Hopf manifolds are LCK and not Vaisman**.

THEOREM: (Ornea-V.)
All complex submanifolds of Vaisman manifolds are Vaisman. All Vaisman manifolds admit a holomorphic embedding to a diagonal Hopf manifold (which is Vaisman, too).

χ-automorphic functions

CLAIM: Conformally equivalent Kähler forms are proportional.

Proof: Let $e^f\omega$ and ω be Kähler forms. Then $0 = d(e^f\omega) = e^f\omega \wedge df$. A multiplication with ω defines an injective map $\Lambda^1(M) \xrightarrow{\wedge\omega} \Lambda^3(M)$, hence $e^f\omega \wedge df = 0$ implies $df = 0$. ■

COROLLARY: Let (M, ω, θ) be an LCK manifold, $(\tilde{M}, \tilde{\omega})$ its Kähler cover. **Then the deck transform group Γ acts on $\tilde{M}, \tilde{\omega}$ by homotheties.** ■

DEFINITION: Denote by $\chi: \Gamma \to \mathbb{R}^{>0}$ the corresponding character, $\gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega}$. A function φ on $\tilde{M}$ is called **χ-automorphic** if $\gamma^*\varphi = \chi(\gamma)\varphi$.

LCK manifolds with potential

DEFINITION: An LCK manifold is called **an LCK manifold with LCK potential** if the Kähler form $\tilde{\omega}$ on $\tilde{M}$ has a χ-automorphic potential, $\tilde{\omega} = dd^c\varphi$, where φ is a χ-automorphic function.

REMARK: A small deformation of an LCK manifold might be non-LCK. A small deformation of Vaisman might be non-Vaisman. **A small deformation of LCK with potential is LCK with potential.**

EXAMPLE: All Hopf manifolds **admit an LCK structure with LCK potential** (Ornea-V.).

THEOREM: (Ornea-V.) A compact manifold M, $\dim_{\mathbb{C}} M > 2$ admits an LCK potential **if and only if M admits a holomorphic embedding to a Hopf manifold.**

REMARK: This property **can be used instead of the definition.**

REMARK: In dimension 2 **this is also true if we assume the GSS conjecture**.

Inoue type LCK manifolds

DEFINITION: A real (p,p)-form A on a complex n-manifold M is called **weakly positive** if $A \wedge \alpha^{n-p}$ is a non-negative top form for any Hermitian form α on M.

DEFINITION: Let A be a weakly positive, non-zero (p,p)-form on a complex manifold M admitting an LCK structure, $\dim_{\mathbb{C}} M > p > 0$. We say that A **consumes the LCK structures** if for any LCK structure (ω, θ) on M, θ is cohomologous to a closed 1-form θ_1 such that $A \wedge \theta_1 = 0$. We say that an LCK manifold is **of Inoue type** if it admits such a form A which is also closed, $dA = 0$.

REMARK: All LCK Inoue surfaces and their generalizations to dim > 2, known as **OT-manifolds** (Oeljeklaus-Toma manifolds) are of this type.

REMARK: All known LCK manifold **belong to one of the following classes:** they are LCK with potential, contain a rational curve, or are of Inoue type.

CLAIM: All LCK complex surfaces **belong to one of these three classes, if GSS conjecture is true.**

Induced globally conformally Kähler (IGCK) submanifolds

DEFINITION: Let (M,θ,ω) be an LCK manifold, and $X \subset M$ a complex subvariety. It is called **induced globally conformally Kähler** (IGCK) if $\theta|_X$ is exact.

EXAMPLE: Let M be a classical Hopf manifold, $M = \mathbb{C}^n\backslash 0/\langle\lambda\,\mathrm{Id}\rangle$, and $E = \frac{\mathbb{C}\backslash 0}{\mathbb{Z}} \subset M$ an elliptic curve obtained from a complex line in $\mathbb{C}^n\backslash 0$. **Clearly, E is Kähler, but the Lee form $\theta = -d\log|z|$ is not exact on E, hence E is not IGCK.**

EXAMPLE: A blow-up of a point in an LCK manifold is again LCK (Tricerri, Vuletescu), and **the exceptional divisor is IGCK, as well as all its submanifolds.**

EXAMPLE: Any Kähler-type submanifold of complex dimension > 1 is IGCK, by Vaisman's theorem.

The fibration theorem

THEOREM: ("Fibration theorem", Ornea-V.-Vuletescu)
Let M be a strict LCK manifold, and $X \subset M$ a submanifold which admits a fibration $\pi: X \to Z$ with complex analytic fibers of positive dimension. Assume that Z is path connected. **Then either X is IGCK, or the fibers of π are IGCK.**

Proof. Step 1: Let $\tilde{M}$ be the minimal Kähler covering of M. Assume that X is not IGCK and the fibers of X are IGCK. **By homotopy lifting lemma, the fibers of π are lifted to $\tilde{M}$.**

Step 2: The deck transform group acts on M by homotheties, hence it takes a fiber of volume V to a fiber of volume cV, where $c \neq 0$. This homothety is non-trivial, because X is not IGCK, hence any connected component $\tilde{X}$ of its preimage is non-compact in $\tilde{M}$. However, **the fibers of $\tilde{X} \to \tilde{Z}$ are homologous, hence they cannot have different volume** (the volume is an integral of the top power of the Kähler form, which is a homology invariant). We have arrined at the contradiction. ■

The main result

THEOREM: Let X be a strict LCK manifold which belongs to one of these three classes: either Inoue type, or has an LCK potential, or has a rational curve, and Y a complex manifold of positive dimension. **Then $X \times Y$ does not admit an LCK structure.**

Proof. Step 1: A product of a manifold and a rational curve **does not admit a strict LCK structure** (Fibration theorem).

Step 2: A product of a complex manifold and a torus does not admit a strict LCK structure, because it admits a continuous isometry, hence it is Vaisman, but a **Vaisman manifold cannot be a product,** because all its subvarieties are tangent to a certain holomorphic foliation.

Step 3: An LCK manifold X with potential always contains an elliptic curve T, hence $M = X \times Y$ contains a subvariety $T \times Y$, which is Kähler (Step 2), hence it is IGCK, **hence Y is IGCK, which contradicts the Fibration theorem,** because M is fibered over X with fibber Y.

Step 4: It remains to consider the case when X is of Inoue type, which is done on the next slide.

Products with an Inoue type manifold

Step 4: Let $n = \dim_{\mathbb{C}} X$, and suppose that $M = X \times Y$ admits an LCK structure (θ, ω). Consider the form $B := \pi_1^* A \wedge \omega^{n-p}$. Then

$$dB = (n-p)\pi_1^* A \wedge \omega^{n-p} \wedge (\pi_1^*\theta_1 + \pi_2^*\theta_2)$$
$$= (n-p)\pi_1^* A \wedge \omega^{n-p} \wedge \pi_2^*\theta_2 = (n-p)B \wedge \pi_2^*\theta_2.$$

Consider the pushforward (that is, the fiberwise integral) $(\pi_2)_* B \in C^\infty(Y)$ of B to Y. Since B is of type (n,n) it follows that

$$(\pi_2)_* B(y) = \int_{\pi_2^{-1}(y)} \left(A \wedge \omega^{n-p}\right)\Big|_{\pi_2^{-1}(y)} = \int_{\pi_2^{-1}(y)} A \wedge \left(\omega^{n-p}\right)\Big|_{\pi_2^{-1}(y)}.$$

Now, from the weak positivity of the (p,p)-form A we infer that the function $(\pi_2)_* B$ is positive and nowhere vanishing. For all $\eta \in \Lambda^* M$, we have

$$(\pi_2)_*(\eta \wedge \pi_2^*\theta_2) = (\pi_2)_*(\eta) \wedge \theta_2.$$

Therefore

$$d(\pi_2)_* B = (\pi_2)_*(dB) = (n-p)(\pi_2)_*(B \wedge \pi_2^*\theta_2) = (n-p)((\pi_2)_* B) \cdot \theta_2.$$

This implies that $\theta_2 = \frac{1}{n-p} d \log((\pi_2)_* B)$ is exact, contradicting the assumption that Y is strictly LCK. ■