Pseudoholomorphic curves with boundary on holomorphic Lagrangian subvarieties

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

REMARK: A hyperkähler manifold M is equipped with 3 symplectic forms ω_I, ω_J, ω_K. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic 2-form on (M,I).** ■

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

COROLLARY: A hyperkähler manifold is **Calabi-Yau,** that is, admits a holomorphic trivialization of its canonical bundle $\Lambda^{\dim_{\mathbb{C}} M,0}(M)$. Indeed, the top power of Ω gives such a trivialization.

THEOREM: (Calabi-Yau)
A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

Calibrations

DEFINITION: (Harvey-Lawson, 1982)
Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

DEFINITION: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

DEFINITION: A **calibration** is a precalibration which is closed.

DEFINITION: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by η** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

H. Blaine Lawson, Jr.,
Berkeley, 1972

F. Reese Harvey,
Berkeley, 1968

Source: George M. Bergman, Berkeley

Calibrations (2)

REMARK: Clearly, for any precalibration η, one has

$$\operatorname{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\operatorname{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, the inequality (*) implies that Z minimizes the Riemannian volume in its homology class.

DEFINITION: A subvariety Z is called **minimal** if for any sufficiently small deformation Z' of Z in class C^1, one has $\operatorname{Vol}(Z') \geqslant \operatorname{Vol}(Z)$.

REMARK: Calibrated subvarieties are obviously minimal.

EXAMPLE: (Wirtenger's inequality)
Let ω be a Kähler form. **Then $\frac{\omega^d}{d!2^d}$ is a calibration which calibrates d-dimensional complex subvarieties.** In patricular, **complex subvarieties in Kähler manifolds are minimal**.

Special Lagrangian subvarieties

DEFINITION: A Calabi-Yau manifold is a Kähler manifold (M, I, ω) admitting a non-degenerate, holomorphic section of canonical bundle $\Phi \in \Gamma(\Lambda^{n,0}(M, I)$ satisfying $|\Phi| = 1$.

REMARK: Let (M, I, Φ, ω) be a Calabi-Yau manifold. **Then $2^{n/2} \operatorname{Re}\Phi$ is a calibration.** Indeed, for any n real tangent vectors $x_1, ..., x_n$ of length 1, one has

$$\operatorname{Re}\Phi(x_1, ..., x_n) = \operatorname{Re}\Phi(x_1^{1,0}, ..., x_n^{1,0}) \leqslant \prod |x_i^{1,0}| = 2^{-n/2}$$

DEFINITION: A special Lagrangian subvariety of (M, I, ω, Φ) is one which is calibrated by $\operatorname{Re}\Phi$.

DEFINITION: A subvariety $X \subset M$ in a symplectic manifold (M, ω) is called **Lagrangian** if $\omega|_X = 0$.

CLAIM: All special Lagrangian subvarieties are Lagrangian.
(see the next slide).

REMARK: Converse is clearly false: any Lagrangian variety can be deformed in such a way that its volume is increased, hence **not all Lagrangian subvarieties are minimal.**

Special Lagrangian subvarieties are Lagrangian

Claim 1: (Harvey-Lawson) All special Lagrangian subvarieties are Lagrangian.

Proof. Step 1: Let V be an n-dimensional Hermitian space equipped with an $(n,0)$-form Φ, $|\Phi| = 1$, and $x_1, ..., x_n \in W$ orthogonal vectors of length 1, such that $2^{n/2} \operatorname{Re}\Phi(x_1, ..., x_n) = 1$. Since $2^{n/2} \operatorname{Re}\Phi$ is a calibration, $2^{n/2} \operatorname{Re}\Phi(x_1^{1,0}, ..., x_n^{1,0})$ takes maximal possible value for all n-tuples $x_1, ..., x_n \in W$.

Step 2: Let $y_1^{1,0}$ be a projection of $x_1^{1,0}$ to an orthogonal complement to $\langle x_2^{1,0}, ..., x_n^{1,0}\rangle$. Then

$$2^{n/2} \operatorname{Re}\Phi(x_1^{1,0}, ..., x_n^{1,0}) = 2^{n/2} \operatorname{Re}\Phi(y_1^{1,0}, ..., x_n^{1,0}). \qquad (**)$$

Step 3: Unless $|y_1^{1,0}| = |x_1^{1,0}|$, one has $|y_1^{1,0}||x_2^{1,0}|...|x_n^{1,0}| < |x_1^{1,0}||x_2^{1,0}|...|x_n^{1,0}|$ giving $2^{n/2} \operatorname{Re}\Phi(y_1^{1,0}, ..., x_n^{1,0}) < 1$, because $\operatorname{comass}(2^{n/2} \operatorname{Re}\Phi) = 1$. Then (**) implies that x_1 is orthogonal to $\langle x_2^{1,0}, ..., x_n^{1,0}\rangle$

Step 4: Since $(x_i^{1,0}, x_j^{1,0}) = \omega(x_i, x_j)$, this implies that $\omega(x_i, x_j) = 0$, and the space generated by x_i is Lagrangian. ■

Special Lagrangian and holomorphic Lagrangian subvarieties

REMARK: Construction of special Lagrangian subvarieties in Calabi-Yau manifolds is a difficult and important problem; **essentially the only way to solve it is to use holomorphically symplectic manifolds.**

DEFINITION: A complex subvariety $X \subset M$ in a holomorphically symplectic manifold (M, I, Ω) is called **Lagrangian** if $\omega|_X = 0$.

Theorem 1: (Harvey-Lawson)
Let (M, I, J, K, g) be a hyperkähler manifold, $X \subset (M, I)$ a holomorphic Lagrangian subvariety, and $L := aJ + bK$ a complex structure, $a^2 + b^2 = 1$, $LI = -LJ$. **Then X is special Lagrangian** with respect to a symplectic form ω_L, where $\omega_L = a\omega_J + b\omega_K$.

(See the proof later in these slides)

Comparison between calibrations

Theorem 1: (Harvey-Lawson)
Let (M, I, J, K, g) be a hyperkähler manifold, $X \subset (M, I)$ a holomorphic Lagrangian subvariety, and $L := aJ + bK$ a complex structure, $a^2 + b^2 = 1$, $LI = -LJ$. **Then X is special Lagrangian** with respect to a symplectic form ω_L, where $\omega_L = a\omega_J + b\omega_K$.

REMARK: This theorem can be deduced from the following result about partial order on calibrations, due to Grantcharov-V.

DEFINITION: Let $\eta, \eta' \in \Lambda^p M$ be two calibrations. We say that **calibration η is smaller than η'**, denoted by $\eta \preceq \eta'$, if $|\eta(x_1, ..., x_p)| \leqslant |\eta'(x_1, ..., x_p)|$.

Theorem 2: Let (M, I, J, K, g) be a hyperkähler manifold, $\Psi := \operatorname{Re} \frac{1}{p!}(\omega_I - \sqrt{-1}\, \omega_K)^p$ the special Lagrangian calibration on (M, J), and $\Psi_I^{p,p}$ its (p, p)-part with respect to I. **Then $\Psi_I^{p,p}$ calibrates holomorphic Lagrangian subvarieties of (M, I). Moreover, $\Psi_I^{p,p} \preceq \Psi$.**

(See the proof later in these slides)

REMARK: Since $\Psi_I^{p,p} \preceq \Psi$, **any holomorphic Lagrangian subvariety of (M, I) is special Lagrangian on (M, J).**

(p,p)-part of a calibration

THEOREM: (Averaging the calibrations)
Let η be a $2p$-form on a complex vector space W, with $\operatorname{comass}(\eta) \leqslant 1$, and $\eta^{p,p} = \operatorname{Av}_\rho \eta$ be the (p,p)-part of η. Then $\operatorname{comass}(\eta^{p,p}) \leqslant 1$. Moreover, **a $2p$-dimensional plane V is a face of $\eta^{p,p}$ if and only if $\rho(t)(V)$ is a face of η for all $t \in \mathbb{R}$.**

Proof: For any decomposable $2p$-vector ξ, its image $\rho(t)(\xi)$ is again decomposable for any t and $|\rho(t)(\xi)| = |\xi|$. Then

$$\eta^{(p,p)}(\xi) = (\operatorname{Av}_\rho(\eta))(\xi) = \frac{1}{2\pi}\int_0^{2\pi} \eta(\rho(t)(\xi)) \leqslant 1$$

since $\eta(\rho(t)\xi) \leqslant 1$ for every t. The equality holds iff $\eta(\rho(t)\xi) = 1$ for every t.
■

COROLLARY: $\eta^{(p,p)} \preceq \eta$.

A holomorphic Lagrangian calibration

Theorem 2 is implied by the following proposition (due to Grantcharov-V.)

PROPOSITION: Let (V^{4p}, I, J, K, g) be a quaternionic Hermitian vector space with fundamental forms $\omega_I, \omega_J, \omega_K$, and $\Psi \in \Lambda^{2p}(V)$ a $2p$-form which is the real part of $\frac{1}{p!}(\omega_I - \sqrt{-1}\,\omega_K)^p$ (it is a $(2p, 0)$-form with respect to J). Denote by $\Psi_I^{p,p}$ the (p, p)-part of Ψ with respect to I. Then $\Psi_I^{p,p}$ has comass 1. Moreover, **a $2p$-dimensional subspace $W \subset V$ is calibrated by $\Psi_I^{p,p}$ if and only if W is $I(W) = W$ and W is calibrated by Ψ.**

Proof. Step 1: The real part of $\frac{1}{p!}(\omega_I - \sqrt{-1}\,\omega_K)^p$ calibrates special Lagrangian subspaces taken with respect to the symplectic form ω_J (Claim 1). Therefore, any face of Ψ is ω_J-Lagrangian.

Step 2: By "averaging the calibration" theorem, $2p$-dimensional plane W is a face of $\Psi_I^{p,p}$ if and only if $\rho(t)(W)$ is a face of Ψ for all $t \in \mathbb{R}$. It follows by taking $t = 0$ that W is ω_J-Lagrangian and by taking $t = \pi/2$ that $I(W)$ is ω_J-Lagrangian, or, equivalently, that W is ω_K-Lagrangian.

Step 3: By Hitchin's lemma below, W is holomorphic Lagrangian on (V, I) if and only if W is ω_J-Lagrangian and $\omega_K K$-Lagrangian. ■

Hitchin's lemma

LEMMA: (Hitchin)
Let (V^{4p}, I, J, K, g) be a quaternionic Hermitian vector space with fundamental forms $\omega_I, \omega_J, \omega_K$, and $W \subset V$ a $2p$-dimensional real vector space. **Then W is holomorphic Lagrangian on (V, I) if and only if W is ω_J-Lagrangian and ω_K-Lagrangian.**

Proof. Step 1: Let L_{ω_J}, Λ_{ω_J} be the Hodge operators, $L_{\omega_J}(\eta) := \eta \wedge \omega_J$, and $\Lambda_{\omega_J} = *L_{\omega_J}*$ its Hermitian adjoint. Consider a decomposable $2p$-vector $\xi \in \Lambda^{2p} V$ which is associated with $W \subset V$. Clearly, **W is Lagrangian with respect to ω_J if and only if $L_{\omega_J}\xi = 0$ and $\Lambda_{\omega_J}\xi = 0$.**

Step 2: The commutator $[L_{\omega_J}, \Lambda_{\omega_K}]$ acts on forms of type (p, q) with respect to I as a multiplication by $(p-q)\sqrt{-1}$, by quaternionic version of Hodge identities: the operators L_{ω_*}, Λ_{ω_*} generate a Lie algebra $\mathfrak{so}(1,4)$. **Then $L_{\omega_J}\xi = \Lambda_{\omega_J}\xi = L_{\omega_K}\xi = \Lambda_{\omega_K}\xi = 0$ implies that ξ is of type (p, p) with respect to I.** ■

Pseudoholomorphic curves in almost complex manifolds

DEFINITION: Let (M, I) be an almost complex manifold. A 2-dimensional compact subvariety $Z \subset M$ (smooth with isolated singularities) is called **a pseudoholomorphic curve** if TZ is I-invariant in all smooth points of Z.

REMARK: If I is a complex structure, **pseudoholomorphic subvarieties are complex curves.**

DEFINITION: Let $Z \subset M$ be a mtric space. **An ε-neighbourhood** $Z(\varepsilon)$ of Z is a union of all ε-balls centered in Z. **Hausdorff metric** d_H on subsets M is defined as follows: $d_H(X, Y)$ is infimum of all ε such that $Y \subset X(\varepsilon)$ and $X \subset Y(\varepsilon)$.

THEOREM: (Gromov)
Let (M, I, g) be an almost complex Hermitian manifold, $C \in \mathbb{R}^{>0}$, and $\mathfrak{S}_C$ the set of all pseudoholomorphic curves S which satisfy $\operatorname{Vol}(S) \leqslant C$, equipped with the Hausdorff metric and induced Hausdorff topology. **Then $\mathfrak{S}_C$ is compact.**

REMARK: For complex curves, this result is due to Bishop (1960-ies).

Pseudoholomorphic curves in symplectic manifolds

DEFINITION: An almost complex structure on a manifold is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$. An almost complex structure **is compatible with a symplectic form** ω if ω is I-invariant, and the symmetric form $g(x,y) = \omega(x, Iy) = \omega(y, Ix)$ is positive definite.

EXERCISE: Prove that **the space of almost complex structures compatible with a given form ω is contractible.**

REMARK: Let $Z \subset M$ be a pseudoholomorphic curve in an almost complex symplectic manifold (M, I, ω). Then $\int_Z \omega = 2\operatorname{Vol}(Z)$.

COROLLARY: (Gromov)
Let (M, I, ω) be an almost complex symplectic manifold, and $\mathfrak{S}$ the set of all pseudoholomorphic curves S, equipped with the Hausdorff metric and induced topology. **Then each connected component of $\mathfrak{S}$ is compact.**

Proof: Since $\int_Z \omega = 2\operatorname{Vol}(Z)$, volume is a cohomological invariant, hence constant on each connected component. Then compactness of each connected component is implied by Gromov's theorem; **indeed, the union of all connected components with $\int_Z \omega \leqslant C$ is compact.** ■

Gromov-Witten invariants

REMARK: Fix a homology class $h \in H_z(M, \mathbb{Z})$, and let $\mathfrak{S}_h$ be the set of all pseudoholomorphic curves S in this homology class. Gromov's theorem implies that **deformation of almost complex structures gives cobordism of the spaces $\mathfrak{S}_h$,** for generic complex structures. On the other hand, the space almost complex structures compatible with a given symplectic structure is connected. This implies that **cobordism invariants of $\mathfrak{S}_h$ such as Euler characteristic and Pontryagin numbers give invariants of symplectic structure on M** ("Gromov-Witten invariants").

Pseudoholomorphic curves with boundary in a Lagrangian subvariety

THEOREM: (Gromov)
Let (M, I, ω) be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and $\mathfrak{S}_L$ the set of all pseudoholomorphic curves S, with boundary on L. Consider $\mathfrak{S}_L$ as a metric space equipped with the Hausdorff metric. **Then each connected component of $\mathfrak{S}$ is compact.**

REMARK: As a motivation for this statement, we remark that (just as it happens in case of curves without boundary), $\operatorname{Vol}(S) = \int_S \omega$ **is a topological invariant,** as the following trivial result implies.

CLAIM: Let (M, I, ω) be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and Z a 2-dimensional singular chain with boundary in L. **Then $\int_S \omega = \langle [S], [\omega] \rangle$, where $[S] \in H_2(M, L)$ and $\omega \in H^2(M, L)$ are the corresponding cohomology classes of a pair (M, L).** ■

REMARK: Just as it happens for curves without boundary, the Euler characteristic and other cobordism invariants of the space $\mathfrak{S}_L(h)$ of all pseudoholomorphic curves with boundary on L for a given homology class $h \in H_2(M, L)$ give an important topological invariant of the Lagrangian subvariety. **It is used to define the Fukaya category of M.**

Pseudoholomorphic curves with boundary in hyperkähler manifolds

The main result of today's talk.

THEOREM: Let (M,I,J,K,g) be a hyperkähler manifold, and $L\subset (M,I)$ a holomorphic Lagrangian subvariety. Consider the set $R=S^1$ of all symplectic structures of form $a\omega_J+b\omega_k,\ a^2+b^2=1$. **Then for all $\omega_1\in R$ except a countable number, there are no pseudoholomorphic curves in (M,ω_1,g) with boundary in L.**

Proof. Step 1: Let S be a 2-dimensional singular chain with boundary on L, and let $h:=[S]\in H_2(M,L)$ be its homology class. Consider the cohomology class $[\omega]\in H^2(M,L)$ of a given $\omega\in\mathbb{R}$, and let $\varphi_h(\omega)=\int_\omega S$ be the corresponding pairing, considered as a function on R. Wirtinger's inequality gives $\int_\omega S\leqslant 2\operatorname{Vol}(S)$, with equality only when S is pseudoholomorphic. This implies that **for any $\omega\in R$ such that S can be represented by a pseudoholomorphic curve, the function $\varphi_h: R\longrightarrow\mathbb{R}$ has a (non-strict) maximum at ω.**

Pseudoholomorphic curves with boundary in hyperkähler manifolds (2)

THEOREM: Let (M, I, J, K, g) be a hyperkähler manifold, and $L \subset (M, I)$ a holomorphic Lagrangian subvariety. Consider the set $R = S^1$ of all symplectic structures of form $a\omega_J + b\omega_k$, $a^2 + b^2 = 1$. **Then for all $\omega_1 \in R$ except a countable number, there are no pseudoholomorphic curves in (M, ω_1, g) with boundary in L.**

Step 2: Let $V \subset H^2(M, L)$ be the space generated by ω_J, ω_K. Extend φ_h to V by the same formula $\varphi_h(\omega) = \int_\omega S$. Since φ_h it is linear on V, its restriction to a circle is either identically zero, or has at most one maximum, denoted by t_h. From Step 1 it follows that **for any complex structure $t' \in R$ not equal to t_h, the class S cannot be represented by a pseudoholomorphic curve with boundary on L.**

Step 3: Let $P \subset R$ be the set of all non-zero maxima $t_h \in R$ for all integer classes $h \in H^2_{\mathbb{Z}}(M, L)$. This set is clearly countable, and **for all $t \notin P$, the corresponding complex structure does not admit pseudoholomorphic curves with boundary in L.** ■

REMARK: You don't even need M or L to be compact for this argument.