# Pseudoholomorphic curves with boundary on holomorphic Lagrangian subvarieties

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**Lagrangian submanifolds and related topics**
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## Complex manifolds

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$ . The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case $I$ is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

**THEOREM:** (Newlander-Nirenberg)
**This definition is equivalent to the usual one.**

## Kähler manifolds

**DEFINITION:** An Riemannian metric $g$ on an almost complex manifiold $M$ is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

**DEFINITION:** The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of $(M, I, g)$.

**THEOREM:** Let $(M, I, g)$ be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure $I$ is integrable, and the Hermitian form $\omega$ is closed.

(ii) One has $\nabla(I) = 0$, where $\nabla$ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

**DEFINITION:** A complex Hermitian manifold $M$ is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form $\omega$ is called **the Kähler class** of $M$. The set of all Kähler classes is called **the Kähler cone**.

## Hyperkähler manifolds

**DEFINITION:** A **hypercomplex manifold** is a manifold $M$ equipped with three complex structure operators $I, J, K$, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

**DEFINITION:** A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric $g$ which is Kähler with respect to $I, J, K$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of $M$.

**REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving $I, J, K$).

## Holomorphically symplectic manifolds

**REMARK:** A hyperkähler manifold $M$ is equipped with 3 symplectic forms $\omega_I$, $\omega_J$, $\omega_K$. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic 2-form on $(M, I)$.** ■

**DEFINITION:** A **holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**COROLLARY:** A hyperkähler manifold is **Calabi-Yau,** that is, admits a holomorphic trivialization of its canonical bundle $\Lambda^{\dim_{\mathbb{C}} M, 0}(M)$. Indeed, the top power of $\Omega$ gives such a trivialization.

**THEOREM:** (Calabi-Yau)
**A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.**

## Calibrations

**DEFINITION:** (Harvey-Lawson, 1982)
Let $W \subset V$ be a $p$-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any $p$-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all $p$-tuples $(v_1, ..., v_p)$ of vectors in $V$ and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

**DEFINITION:** A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

**DEFINITION:** A **calibration** is a precalibration which is closed.

**DEFINITION:** Let $\eta$ be a $k$-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a $k$-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of $Z$ is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over $Z$). We say that $Z$ is **calibrated by $\eta$** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration $\eta$.

H. Blaine Lawson, Jr.,
Berkeley, 1972

F. Reese Harvey,
Berkeley, 1968

Source: George M. Bergman, Berkeley

## Calibrations (2)

**REMARK:** Clearly, for any precalibration $\eta$, one has

$$\operatorname{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\operatorname{Vol}(Z)$ denotes the Riemannian volume of a compact $Z$, and the equality happens iff $Z$ is calibrated by $\eta$. If, in addition, $\eta$ is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, the inequality (*) implies that $Z$ minimizes the Riemannian volume in its homology class.

**DEFINITION:** A subvariety $Z$ is called **minimal** if for any sufficiently small deformation $Z'$ of $Z$ in class $C^1$, one has $\operatorname{Vol}(Z') \geqslant \operatorname{Vol}(Z)$.

**REMARK: Calibrated subvarieties are obviously minimal.**

**EXAMPLE:** (Wirtenger's inequality)
Let $\omega$ be a Kähler form. **Then $\frac{\omega^d}{d!2^d}$ is a calibration which calibrates $d$-dimensional complex subvarieties.** In patricular, **complex subvarieties in Kähler manifolds are minimal**.

## Special Lagrangian subvarieties

**DEFINITION: A Calabi-Yau manifold** is a Kähler manifold $(M, I, \omega)$ admitting a non-degenerate, holomorphic section of canonical bundle $\Phi \in \Gamma(\Lambda^{n,0}(M, I)$ satisfying $|\Phi| = 1$.

**REMARK:** Let $(M, I, \Phi, \omega)$ be a Calabi-Yau manifold. **Then $2^{n/2} \operatorname{Re} \Phi$ is a calibration.** Indeed, for any $n$ real tangent vectors $x_1, ..., x_n$ of length 1, one has

$$\operatorname{Re} \Phi(x_1, ..., x_n) = \operatorname{Re} \Phi(x_1^{1,0}, ..., x_n^{1,0}) \leqslant \prod |x_i^{1,0}| = 2^{-n/2}$$

**DEFINITION: A special Lagrangian subvariety** of $(M, I, \omega, \Phi)$ is one which is calibrated by $\operatorname{Re} \Phi$.

**DEFINITION:** A subvariety $X \subset M$ in a symplectic manifold $(M, \omega)$ is called **Lagrangian** if $\omega|_X = 0$.

**CLAIM: (Harvey-Lawson)**
**All special Lagrangian subvarieties are Lagrangian.**

**REMARK:** Converse is clearly false: any Lagrangian variety can be deformed in such a way that its volume is increased, hence **not all Lagrangian subvarieties are minimal.**

## Special Lagrangian and holomorphic Lagrangian subvarieties

**REMARK:** Construction of special Lagrangian subvarieties in Calabi-Yau manifolds is a difficult and important problem; **essentially the only way to solve it is to use holomorphically symplectic manifolds.**

**DEFINITION:** A complex subvariety $X \subset M$ in a holomorphically symplectic manifold $(M, I, \Omega)$ is called **holomorphic Lagrangian** if $\Omega|_X = 0$.

**Theorem: (Harvey-Lawson)**
Let $(M, I, J, K, g)$ be a hyperkähler manifold, $X \subset (M, I)$ a holomorphic Lagrangian subvariety, and $J_1 := aJ + bK$ a complex structure, $a^2 + b^2 = 1$, $J_1 I = -J_1 I$. **Then $X$ is special Lagrangian** with respect to a symplectic form $\omega_{J_1}$, where $\omega_{J_1} = a\omega_J + b\omega_K$.

## Pseudoholomorphic curves in almost complex manifolds

**DEFINITION:** Let $(M, I)$ be an almost complex manifold. A 2-dimensional compact subvariety $Z \subset M$ (smooth with isolated singularities) is called **a pseudoholomorphic curve** if $TZ$ is $I$-invariant in all smooth points of $Z$.

**REMARK:** If $I$ is a complex structure, **pseudoholomorphic subvarieties are complex curves.**

**DEFINITION:** Let $Z \subset M$ be a mtric space. **An $\varepsilon$-neighbourhood** $Z(\varepsilon)$ of $Z$ is a union of all $\varepsilon$-balls centered in $Z$. **Hausdorff metric** $d_H$ on subsets $M$ is defined as follows: $d_H(X, Y)$ is infimum of all $\varepsilon$ such that $Y \subset X(\varepsilon)$ and $X \subset Y(\varepsilon)$.

**THEOREM: (Gromov)**
Let $(M, I, g)$ be an almost complex Hermitian manifold, $C \in \mathbb{R}^{>0}$, and $\mathfrak{S}_C$ the set of all pseudoholomorphic curves $S$ which satisfy $\operatorname{Vol}(S) \leqslant C$, equipped with the Hausdorff metric and induced Hausdorff topology. **Then $\mathfrak{S}_C$ is compact.**

**REMARK:** For complex curves, this result is due to Bishop (1960-ies).

## Pseudoholomorphic curves in symplectic manifolds

**DEFINITION: An almost complex structure** on a manifold is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$. An almost complex structure **is compatible with a symplectic form** $\omega$ if $\omega$ is $I$-invariant, and the symmetric form $g(x,y) = \omega(x,Iy) = \omega(y,Ix)$ is positive definite.

**EXERCISE:** Prove that **the space of almost complex structures compatible with a given form $\omega$ is contractible.**

**REMARK:** Let $Z \subset M$ be a pseudoholomorphic curve in an almost complex symplectic manifold $(M,I,\omega)$. Then $\int_Z \omega = 2\operatorname{Vol}(Z)$.

**COROLLARY: (Gromov)**
Let $(M,I,\omega)$ be an almost complex symplectic manifold, and $\mathfrak{S}$ the set of all pseudoholomorphic curves $S$, equipped with the Hausdorff metric and induced topology. **Then each connected component of $\mathfrak{S}$ is compact.**

**Proof:** Since $\int_Z \omega = 2\operatorname{Vol}(Z)$, volume is a cohomological invariant, hence constant on each connected component. Then compactness of each connected component is implied by Gromov's theorem; **indeed, the union of all connected components with $\int_Z \omega \leqslant C$ is compact.** ■

## Gromov-Witten invariants

**REMARK:** Fix a homology class $h \in H_z(M, \mathbb{Z})$, and let $\mathfrak{S}_h$ be the set of all pseudoholomorphic curves $S$ in this homology class. Gromov's theorem implies that **deformation of almost complex structures gives cobordism of the spaces $\mathfrak{S}_h$**, for generic complex structures. On the other hand, the space almost complex structures compatible with a given symplectic structure is connected. This implies that **cobordism invariants of $\mathfrak{S}_h$ such as Euler characteristic and Pontryagin numbers give invariants of symplectic structure on $M$** ("Gromov-Witten invariants").

## Pseudoholomorphic curves with boundary in a Lagrangian subvariety

**THEOREM: (Gromov)**
Let $(M, I, \omega)$ be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and $\mathfrak{S}_L$ the set of all pseudoholomorphic curves $S$, with boundary on $L$. Consider $\mathfrak{S}_L$ as a metric space equipped with the Hausdorff metric. **Then each connected component of $\mathfrak{S}_L$ is compact.**

**REMARK:** As a motivation for this statement, we remark that (just as it happens in case of curves without boundary), $\operatorname{Vol}(S) = \int_S \omega$ **is a topological invariant,** as the following trivial result implies.

**CLAIM:** Let $(M, I, \omega)$ be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and $Z$ a 2-dimensional singular chain with boundary in $L$. **Then $\int_S \omega = \langle [S], [\omega] \rangle$, where $[S] \in H_2(M, L)$ and $\omega \in H^2(M, L)$ are the corresponding cohomology classes of a pair $(M, L)$.** ■

**REMARK:** Just as it happens for curves without boundary, the Euler characteristic and other cobordism invariants of the space $\mathfrak{S}_L(h)$ of all pseudoholomorphic curves with boundary on $L$ for a given homology class $h \in H_2(M, L)$ give an important topological invariant of the Lagrangian subvariety. **It is used to define the Fukaya category of $M$.**

## Pseudoholomorphic curves with boundary in hyperkähler manifolds

The main result of today's talk.

**THEOREM:** Let $(M, I, J, K, g)$ be a hyperkähler manifold, and $L \subset (M, I)$ a holomorphic Lagrangian subvariety. Consider the set $R = S^1$ of all symplectic structures of form $a\omega_J + b\omega_k$, $a^2 + b^2 = 1$. **Then for all $\omega_1 \in R$ except a countable number, there are no pseudoholomorphic curves in $(M, \omega_1, g)$ with boundary in $L$.**

**Proof. Step 1:** Let $S$ be a 2-dimensional singular chain with boundary on $L$, and let $h := [S] \in H_2(M, L)$ be its homology class. Consider the cohomology class $[\omega] \in H^2(M, L)$ of a given $\omega \in \mathbb{R}$, and let $\varphi_h(\omega) = \int_\omega S$ be the corresponding pairing, considered as a function on $R$. Wirtinger's inequality gives $\int_\omega S \leqslant 2\,\mathrm{Vol}(S)$, with equality only when $S$ is pseudoholomorphic. This implies that **for any $\omega \in R$ such that $S$ can be represented by a pseudoholomorphic curve, the function $\varphi_h : R \longrightarrow \mathbb{R}$ has a (non-strict) maximum at $\omega$.**

## Pseudoholomorphic curves with boundary in hyperkähler manifolds (2)

**THEOREM:** Let $(M, I, J, K, g)$ be a hyperkähler manifold, and $L \subset (M, I)$ a holomorphic Lagrangian subvariety. Consider the set $R = S^1$ of all symplectic structures of form $a\omega_J + b\omega_k$, $a^2 + b^2 = 1$. **Then for all $\omega_1 \in R$ except a countable number, there are no pseudoholomorphic curves in $(M, \omega_1, g)$ with boundary in $L$.**

**Step 2:** Let $V \subset H^2(M, L)$ be the space generated by $\omega_J, \omega_K$. Extend $\varphi_h$ to $V$ by the same formula $\varphi_h(\omega) = \int_\omega S$. Since $\varphi_h$ it is linear on $V$, its restriction to a circle is either identically zero, or has at most one maximum, denoted by $t_h$. From Step 1 it follows that **for any complex structure $t' \in R$ not equal to $t_h$, the class $S$ cannot be represented by a pseudoholomorphic curve with boundary on $L$.**

**Step 3:** Let $P \subset R$ be the set of all non-zero maxima $t_h \in R$ for all integer classes $h \in H^2_{\mathbb{Z}}(M, L)$. This set is clearly countable, and **for all $t \notin P$, the corresponding complex structure does not admit pseudoholomorphic curves with boundary in $L$.** ■

**REMARK:** You don't even need $M$ or $L$ to be compact for this argument.