Pseudoholomorphic curves with boundary on holomorphic Lagrangian subvarieties and formality in the Fukaya category

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

REMARK: A hyperkähler manifold M is equipped with 3 symplectic forms ω_I, ω_J, ω_K. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic 2-form on (M, I).** ■

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

COROLLARY: A hyperkähler manifold is **Calabi-Yau,** that is, admits a holomorphic trivialization of its canonical bundle $\Lambda^{\dim_{\mathbb{C}} M, 0}(M)$. Indeed, the top power of Ω gives such a trivialization.

THEOREM: (Calabi-Yau)
A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

Calibrations

DEFINITION: (Harvey-Lawson, 1982)
Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\operatorname{Vol}(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let **comass** $\operatorname{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, ..., v_p)}{|v_1||v_2|...|v_p|}$, for all p-tuples $(v_1, ..., v_p)$ of vectors in V and **face** be the set of planes $W \subset V$ where $\frac{\eta}{\operatorname{Vol}(W)} = \operatorname{comass}(\eta)$.

DEFINITION: A **precalibration** on a Riemannian manifold is a differential form with comass $\leqslant 1$ everywhere.

DEFINITION: A **calibration** is a precalibration which is closed.

DEFINITION: Let η be a k-dimensional precalibration on a Riemannian manifold, and $Z \subset M$ a k-dimensional subvariety (we always assume that the Hausdorff dimension of the set of singular points of Z is $\leqslant k-2$, because in this case a compactly supported differential form can be integrated over Z). We say that Z is **calibrated by η** if at any smooth point $z \in Z$, the space $T_z Z$ is a face of the precalibration η.

H. Blaine Lawson, Jr.,
Berkeley, 1972

F. Reese Harvey,
Berkeley, 1968

Source: George M. Bergman, Berkeley

Calibrations (2)

REMARK: Clearly, for any precalibration η, one has

$$\operatorname{Vol}(Z) \geqslant \int_Z \eta, \qquad (*)$$

where $\operatorname{Vol}(Z)$ denotes the Riemannian volume of a compact Z, and the equality happens iff Z is calibrated by η. If, in addition, η is closed, the number $\int_Z \eta$ is a cohomological invariant. Then, the inequality (*) implies that Z minimizes the Riemannian volume in its homology class.

DEFINITION: A subvariety Z is called **minimal** if for any sufficiently small deformation Z' of Z in class C^1, one has $\operatorname{Vol}(Z') \geqslant \operatorname{Vol}(Z)$.

REMARK: **Calibrated subvarieties are obviously minimal.**

EXAMPLE: (Wirtenger's inequality)
Let ω be a Kähler form. **Then $\frac{\omega^d}{d!2^d}$ is a calibration which calibrates d-dimensional complex subvarieties.** In patricular, **complex subvarieties in Kähler manifolds are minimal**.

Special Lagrangian subvarieties

DEFINITION: A Calabi-Yau manifold is a Kähler manifold (M, I, ω) admitting a non-degenerate, holomorphic section of canonical bundle $\Phi \in \Gamma(\Lambda^{n,0}(M, I)$ satisfying $|\Phi| = 1$.

REMARK: Let (M, I, Φ, ω) be a Calabi-Yau manifold. **Then $2^{n/2} \operatorname{Re} \Phi$ is a calibration.** Indeed, for any n real tangent vectors $x_1, ..., x_n$ of length 1, one has

$$\operatorname{Re} \Phi(x_1, ..., x_n) = \operatorname{Re} \Phi(x_1^{1,0}, ..., x_n^{1,0}) \leqslant \prod |x_i^{1,0}| = 2^{-n/2}$$

DEFINITION: A special Lagrangian subvariety of (M, I, ω, Φ) is one which is calibrated by $\operatorname{Re} \Phi$.

CLAIM: (Harvey-Lawson)
All special Lagrangian subvarieties are Lagrangian.

REMARK: Converse is clearly false: any Lagrangian variety can be deformed in such a way that its volume is increased, hence **not all Lagrangian subvarieties are minimal.**

Special Lagrangian and holomorphic Lagrangian subvarieties

REMARK: Construction of families of special Lagrangian subvarieties in Calabi-Yau manifolds is a difficult and important problem; **essentially the only way to solve it is to use holomorphically symplectic manifolds.**

DEFINITION: A complex subvariety $X \subset M$ in a holomorphically symplectic manifold (M, I, Ω) is called **holomorphic Lagrangian** if $\Omega|_X = 0$.

Theorem: (Harvey-Lawson)
Let (M, I, J, K, g) be a hyperkähler manifold, $X \subset (M, I)$ a holomorphic Lagrangian subvariety, and $J_1 := aJ + bK$ a complex structure, $a^2 + b^2 = 1$, $J_1 I = -J_1 I$. **Then X is special Lagrangian** with respect to a symplectic form ω_{J_1}, where $\omega_{J_1} = a\omega_J + b\omega_K$.

Pseudoholomorphic curves in almost complex manifolds

DEFINITION: Let (M, I) be an almost complex manifold. A 2-dimensional compact subvariety $Z \subset M$ (smooth with isolated singularities) is called **a pseudoholomorphic curve** if TZ is I-invariant in all smooth points of Z.

REMARK: If I is a complex structure, **pseudoholomorphic subvarieties are complex curves.**

DEFINITION: Let $Z \subset M$ be a metric space. **An ε-neighbourhood** $Z(\varepsilon)$ of Z is a union of all ε-balls centered in Z. **Hausdorff metric** d_H on subsets M is defined as follows: $d_H(X, Y)$ is infimum of all ε such that $Y \subset X(\varepsilon)$ and $X \subset Y(\varepsilon)$.

THEOREM: (Gromov)
Let (M, I, g) be an almost complex Hermitian manifold, $C \in \mathbb{R}^{>0}$, and $\mathfrak{S}_C$ the set of all pseudoholomorphic curves S which satisfy $\operatorname{Vol}(S) \leqslant C$, equipped with the Hausdorff metric and induced Hausdorff topology. **Then $\mathfrak{S}_C$ is compact.**

REMARK: For complex curves, this result is due to Bishop (1960-ies).

Pseudoholomorphic curves in symplectic manifolds

DEFINITION: **An almost complex structure** on a manifold is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$. An almost complex structure **is compatible with a symplectic form ω** if ω is I-invariant, and the symmetric form $g(x,y) = \omega(x, Iy) = \omega(y, Ix)$ is positive definite.

EXERCISE: Prove that **the space of almost complex structures compatible with a given form ω is contractible.**

REMARK: Let $Z \subset M$ be a pseudoholomorphic curve in an almost complex symplectic manifold (M, I, ω). Then $\int_Z \omega = 2\operatorname{Vol}(Z)$.

COROLLARY: (Gromov)
Let (M, I, ω) be an almost complex symplectic manifold, and $\mathfrak{S}$ the set of all pseudoholomorphic curves S, equipped with the Hausdorff metric and induced topology. **Then each connected component of $\mathfrak{S}$ is compact.**

Proof: Since $\int_Z \omega = 2\operatorname{Vol}(Z)$, volume is a cohomological invariant, hence constant on each connected component. Then compactness of each connected component is implied by Gromov's theorem; **indeed, the union of all connected components with $\int_Z \omega \leqslant C$ is compact.** ■

Pseudoholomorphic curves with boundary in a Lagrangian subvariety

THEOREM: (Gromov)
Let (M, I, ω) be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and $\mathfrak{S}_L$ the set of all pseudoholomorphic curves S, with boundary on L. Consider $\mathfrak{S}_L$ as a metric space equipped with the Hausdorff metric. **Then each connected component of $\mathfrak{S}_L$ is compact.**

REMARK: As a motivation for this statement, we remark that (just as it happens in case of curves without boundary), $\operatorname{Vol}(S) = \int_S \omega$ **is a topological invariant,** as the following trivial result implies.

CLAIM: Let (M, I, ω) be an almost complex symplectic manifold, $L \subset M$ a Lagrangian subvariety, and Z a 2-dimensional singular chain with boundary in L. **Then $\int_S \omega = \langle [S], [\omega] \rangle$, where $[S] \in H_2(M, L)$ and $\omega \in H^2(M, L)$ are the corresponding (co-)homology classes in (co-)homology of a pair (M, L).** ■

REMARK: Just as it happens for curves without boundary, the Euler characteristic and other cobordism invariants of the space $\mathfrak{S}_L(h)$ of all pseudoholomorphic curves with boundary on L for a given homology class $h \in H_2(M, L)$ give an important topological invariant of the Lagrangian subvariety. **It is used to define the Fukaya category of M.**

Pseudoholomorphic curves with boundary in hyperkähler manifolds

The main result of today's talk.

THEOREM: Let (M, I, J, K, g) be a hyperkähler manifold, and $L \subset (M, I)$ a holomorphic Lagrangian subvariety. Consider the set $R = S^1$ of all symplectic structures of form $a\omega_J + b\omega_k$, $a^2 + b^2 = 1$. **Then for all $\omega_1 \in R$ except a countable number, there are no pseudoholomorphic curves in (M, ω_1, g) with boundary in L.**

Proof. Step 1: Let S be a 2-dimensional singular chain with boundary on L, and let $h := [S] \in H_2(M, L)$ be its homology class. Consider the cohomology class $[\omega] \in H^2(M, L)$ of a given $\omega \in \mathbb{R}$, and let $\varphi_h(\omega) = \int_\omega S$ be the corresponding pairing, considered as a function on R. Wirtinger's inequality gives $\int_\omega S \leqslant 2\operatorname{Vol}(S)$, with equality only when S is pseudoholomorphic. This implies that **for any $\omega \in R$ such that S can be represented by a pseudoholomorphic curve, the function $\varphi_h : R \longrightarrow \mathbb{R}$ has a (non-strict) maximum at ω.**

Pseudoholomorphic curves with boundary in hyperkähler manifolds (2)

THEOREM: Let (M, I, J, K, g) be a hyperkähler manifold, and $L \subset (M, I)$ a holomorphic Lagrangian subvariety. Consider the set $R = S^1$ of all symplectic structures of form $a\omega_J + b\omega_k$, $a^2 + b^2 = 1$. **Then for all $\omega_1 \in R$ except a countable number, there are no pseudoholomorphic curves in (M, ω_1, g) with boundary in L.**

Step 2: Let $V \subset H^2(M, L)$ be the space generated by ω_J, ω_K. Extend φ_h to V by the same formula $\varphi_h(\omega) = \int_\omega S$. Since φ_h it is linear on V, its restriction to a circle is either identically zero, or has at most one maximum, denoted by t_h. From Step 1 it follows that **for any complex structure $t' \in R$ not equal to t_h, the class S cannot be represented by a pseudoholomorphic curve with boundary on L.**

Step 3: Let $P \subset R$ be the set of all non-zero maxima $t_h \in R$ for all integer classes $h \in H^2_{\mathbb{Z}}(M, L)$. This set is clearly countable, and **for all $t \notin P$, the corresponding complex structure does not admit pseudoholomorphic curves with boundary in L.** ∎

REMARK: You don't even need M or L to be compact for this argument.

A_∞-categories

An A_∞ **category** $\mathcal{A}$ consists of the following data:

- A collection of objects $\mathcal{O}b\mathcal{A}$.
- For each pair of objects $A, B \in \mathcal{O}b\mathcal{A}$, a graded vector space $Hom_{\mathcal{A}}(A, B)$.
- For each $(k+1)$-tuple of objects $A_0, \ldots, A_k$, $k \geqslant 0$, a multilinear map

$$\mu_k^{\mathcal{A}} : \bigotimes_{i=1}^k Hom_{\mathcal{A}}(A_{i-1}, A_i) \to Hom_{\mathcal{A}}(A_0, A_k)[2-k].$$

satisfying relation

$$\sum_{m_1+\cdots+m_l=k} \mu_l^{\mathcal{B}}(f_{m_1}(\alpha_1, \ldots, \alpha_{m_1}), \ldots, f_{m_l}(\alpha_{k-m_l+1}, \ldots, \alpha_k)) =$$
$$= \sum_{k_1+k_2=k+1} (-1)^\star f_{k_1}(\alpha_1, \ldots, \alpha_{i-1}, \mu_{k_2}^{\mathcal{A}}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \ldots, \alpha_k),$$

where $\star = q - 1 + \sum_{j=1}^{q-1} |\alpha_j|$.

EXAMPLE: Fukaya category, [FOOO] on a symplectic manifold is a category where objects are Lagrangian subvarieties, and the maps μ_i are defined as integrals of certain cohomology classes over the moduli of pseudoholomorphic disks with boundary on these Lagrangian varieties. By Gromov's compactness, μ_i are invariant under the Hamiltonian isotopy of the Lagrangian varieties.

Formality

DEFINITION: Let $\mathcal{A}$ be an A_∞ category. The **associated cohomological category** $H(\mathcal{A})$ has the same objects, its morphism spaces are given by

$$Hom_{H(\mathcal{A})}(A,B) = H^*(Hom_{\mathcal{A}}(A,B), \mu_1^{\mathcal{A}}),$$

and the composition of morphisms is given by

$$[a_2] \circ [a_1] = (-1)^{|a_1|}[\mu_2(a_1,a_2)]. \quad (1)$$

We consider $H(\mathcal{A})$ as an A_∞ category, which h as all operations zero except for $k=2$.

DEFINITION: An A_∞ functor $f:\mathcal{A}\to\mathcal{B}$ induces a functor $H(f): H(\mathcal{A}) \to H(B)$. It is called **quasi-isomorphism** if $H(f)$ is an equivalence of categories. An A_∞-category $\mathcal{A}$ is called **formal** if it is quasi-isomorphic to $H(\mathcal{A})$.

EXAMPLE: Let M be an almost complex symplectic manifold, and $\{L_i\}$ a collection of Lagrangian subvarieties. Assume that all pseudoholomorphic disks with boundaries on $\{L_i\}$ are trivial. **Then the corresponding Fukaya category is formal**.

Special Lagrangian manifolds and Fukaya category

DEFINITION: Let $\mathcal{L}$ be the (∞-dimensional) space of Lagrangian submanifolds equipped with a flat $U(1)$-bundle. The manifold $\mathcal{L}$ is symplectic: indeed, the tangent space of the space of flat $U(1)$ bundles on L in the space $\Lambda^1_{cl}(L)$ of closed 1-forms on L, and the tangent space to the space of Lagrangian submanifolds is also $\Lambda^1_{cl}(L)$.

CLAIM: (R. Thomas, S.-T. Yau)
Consider the Hamiltonian diffeomorphism group Ham acting on the space $\mathcal{L}$ of Lagrangian submanifolds in a Calabi-Yau manifold. **Then its moment map is given by the mean curvature,** and the zeros of the moment map are special Lagrangian submanifolds.

DEFINITION: A Lagrangian manifold is called **stable** if its Ham-orbit contains a special Lagrangian manifold.

CONJECTURE: (R. Thomas, S.-T. Yau)
Any Lagrangian manifold can be obtained as a connected sum of stable special Lagrangian submanifolds.

If we believe in this conjecture, **to study the Fukaya category on a Calabi-Yau manifold it would suffice to consider only special Lagrangian submanifolds.**

Special Lagrangian manifolds and formality

THEOREM: Let $\{L_i\}$ be a collection of holomorphic Lagrangian submanifolds in a hyperkähler manifold (M, I, J, K), and $\omega_1 := a\omega_J + b\omega_K$ a general linear combination. **Then the corresponding Fukaya category is formal.**

This theorem follows from

THEOREM: Let $\{L_i\}$ be a collection of special Lagrangian submanifolds, and $J_1 := aJ + bK$ a complex structure with a, b general. **Then any J_1-holomorphic disks with boundary on $\cup L_i$ are constant.**

Proof: It would follow by the same argument as above, if we prove the following highly non-trivial result.

THEOREM: Let $\{L_i\}$ be a collection of real analytic Lagrangian submanifolds in a compact real analytic symplectic manifold M, and $\mathcal{S}$ the set of homotopy classes of maps from disk to M with boundary on $\cup L_i$. **Then $\mathcal{S}$ is countable.**