The Lee-Gauduchon cone on non-Kähler complex manifolds

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dd^c-lemma

DEFINITION: Let M be a complex manifold, and $I: TM \to TM$ its complex structure operator. **The twisted differential** $d^c := IdI^{-1}: \Lambda^*(M) \to \Lambda^{*+1}(M)$, where I acts on 1-forms as an operator dual to $I: TM \to TM$, and on the rest of differential forms multiplicatively.

REMARK: Consider the Hodge decomposition of the de Rham differential, $d = \partial + \overline{\partial}$, where $\partial: \Lambda^{p,q}(M,I) \to \Lambda^{p+1,q}(M,I)$ and $\overline{\partial}: \Lambda^{p,q}(M,I) \to \Lambda^{p+1,q}(M,I)$. **Then $d = \operatorname{Re}\partial$ and $d^c = \operatorname{Im}\partial$.** Also, $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

THEOREM: **The following statements are equivalent.** 1. I is integrable. 2. $\partial^2 = 0$. 3. $\overline{\partial}^2 = 0$. 4. $dd^c = -d^c d$ 5. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

THEOREM: (**dd^c-lemma**) Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.

(1) η is an exact (p,q)-form.
(2) η is d-exact, d^c-closed.
(3) η is $\overline{\partial}$-closed, ∂-exact.
(4) η is ∂-closed, $\overline{\partial}$-exact.

Then $\eta \in \operatorname{im} dd^c$.

REMARK: **These 4 statements are equivalent on any complex manifold** (Deligne-Griffiths-Morgan-Sullivan; proven by diagram-chasing).

Bott-Chern cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}M)$. Then $H^{p,q}_{BC}(M)$ is called **the Bott–Chern cohomology** of M.

REMARK: A (p,q)-form η is closed if and only if $\partial\eta = \overline{\partial}\eta = 0$. Using $2\sqrt{-1}\,\partial\overline{\partial} = dd^c$, **we could define the Bott–Chern cohomology $H^*_{BC}(M)$ as $H^*_{BC}(M) := \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}$.**

REMARK: There are natural (and functorial) maps from the Bott–Chern cohomology to the Dolbeault cohomology $H^*(\Lambda^{*,*}M, \overline{\partial})$ and to the de Rham cohomology, **but no morphisms between the de Rham and the Dolbeault cohomology.**

REMARK: Good things about the Bott-Chern cohomology: there are functorial maps from Bott-Chern to **both** Dolbeault and de Rham cohomology.

REMARK: On any manifold where dd^c-lemma is true, **Bott-Chern cohomology is equal to Dolbeault and de Rham cohomology.**

Aeppli cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{AE}(M)$ the space of dd^c-closed (p,q)-forms modulo $\partial(\Lambda^{p-1,q}M) + \overline{\partial}(\Lambda^{p,q-1}M)$. Then $H^{p,q}_{AE}(M)$ is called **the Aeppli cohomology** of M.

THEOREM: (A. Aeppli)
Let M be a compact complex n-manifold. Then **the Aeppli cohomology is finite-dimensional.** Moreover, the natural pairing

$$H^{p,q}_{BC}(M) \times H^{n-p,n-q}_{AE}(M) \to H^{2n}(M) = \mathbb{C},$$

taking x, y to $\int_M x \wedge y$ **is non-degenerate and identifies $H^{p,q}_{BC}(M)$ with the dual $H^{n-p,n-q}_{AE}(M)^*$.**

REMARK: Math Genealogy knows 3 person called Aeppli: Alfred Aeppli (ETH Zürich, 1924, student of George Pólya and Hermann Weyl), Alfred Aeppli (ETH Zürich, 1956, student of Beno Eckmann and Heinz Hopf), and Hans Aeppli (1980, student of Hans Storrer). *The second Alfred Aeppli was the one responsible for Aeppli cohomology.*

Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n-manifold is called **Gauduchon** if $dd^c(\omega^{n-1}) = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique in any given conformal class, up to a constant multiplier.

REMARK: This is one of **very few statements** which is valid (and can be applied) to all compact complex manifolds.

REMARK: This is very useful, because allows to define **a degree** of a holomorphic bundle, define stability, and prove a **non-Kähler version of Donaldson-Uhlenbeck-Yau therem.**

The Gauduchon cone

DEFINITION: Let M be a complex manifold, and ω a Gauduchon metric. **A Gauduchon form** of M is ω^{n-1}.

REMARK: A form is positive (evalues positively on all complex $(n-1, n-1)$-spaces) if and only if it is $n-1$ power of a Hermitian form. Therefore, **the space of Gauduchon forms is a convex cone in $\Lambda^{n-1,n-1}(M, \mathbb{R})$.**

DEFINITION: The Gauduchon cone of a compact complex n-manifold is the set of all classes $\omega^{n-1} \in H^{n-1,n-1}_{AE}(M)$ of all Gauduchon forms.

DEFINITION: Recall that **pseudoeffective cone** $P \subset H^{1,1}_{BC}(M)$ is the cone of all Bott-Chern classes of all positive, closed (1,1)-currents.

THEOREM: (Lamari)
The Gauduchon cone is dual to the pseudoeffective cone.

Proof: Follows from the Hahn-Banach theorem (the proof is given later).

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M} \left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

REMARK: The pairing between forms and currents is denoted as $\alpha, \tau \mapsto \int_M \alpha \wedge \tau$. Using this notation, **we interpret k forms on n-manifold as k-currents,** that is, as functionals on $n-k$-forms.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact).

CLAIM: De Rham differential is continuous on currents, and the Poincaré lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

CLAIM: The Poincaré and Poincaré-Dolbeault-Grothendieck lemma hold on (p,q)-currents, and **the d- and $\overline{\partial}$-cohomology are the same as for forms.**

REMARK: Integration currents of complex submanifolds (or subvarieties) **are closed (p,p)-currents.**

DEFINITION: **A cone of positive (1,1)-currents** is generated by $\alpha u \wedge I(u)$ where u is a real 1-form and α a measure; equivalently, positive (1,1)-currents are currents ξ such that $\int_M \xi \wedge \tau \geqslant 0$ for any positive $(n-1,n-1)$-form τ.

Hahn-Banach theorem

THEOREM: Let V be a locally convex topological vector space, $W \subset V$ a closed subspace, and $A \subset V$ an open, convex subset, not intersecting A. **Then there exists a continuous linear functional $\xi \in V^*$ vanishing on W and positive on A.**

COROLLARY: Let $\eta \in H^{n-1,n-1}_{AE}(M,\mathbb{R})$ be a cohomology class. Then η cannot be represented by a Gauduchon form **if and only if there exists a (1,1)-current $\xi \in D^{1,1}$ which vanishes on dd^c-closed $(n-1,n-1)$-forms and evaluates positively on positive $(n-1,n-1)$-forms.**

Proof: Apply Hahn-Banach to W the space of dd^c-closed $(n-1,n-1)$-forms and A the space of positive $(n-1,n-1)$-forms. ■

REMARK: The current ξ is positive, because it evaluates positively on positive $(n-1,n-1)$-forms. Also, $\langle d\xi, u\rangle = \langle \xi, du\rangle = 0$ for any $(2n-3)$-form u, because the $(n-1,n-1)$-part of a closed form is dd^c-closed. Therefore, ξ is closed. **This proves Lamari's theorem.**

An exact sequence in Bott-Chern cohomology

CLAIM: Let $\tau: H^{1,1}_{BC}(M){\to}H^2(M)$ be the tautological map, and $\operatorname{Re}\circ d$ the composition of d and taking the real part. Denote by $H^{1,0}_d(M)$ the space of closed holomorphic forms on M. **Then the sequence**

$$0{\to}H^{1,0}_d(M)\oplus\overline{H^{1,0}_d(M)}{\to}H^1(M)\xrightarrow{d^c}H^{1,1}_{BC}(M,\mathbb{R})\xrightarrow{\tau}H^2(M)\quad(*)$$

is exact.

Proof. Step 1: This sequence is clearly exact in the first term: an exact holomorphic form is a differential of a global holomorphic function on M, and all such functions are constant.

Step 2: To prove that it is exact in the second term, let x be a closed 1-form, and let $[x]\in H^1(M)$ be its cohomology class. The cohomology class of $d^c x$ vanishes in $H^{1,1}_{BC}(M,\mathbb{R})$ if and only if $d^c x = dd^c f$, for some function $f\in C^\infty M$. However, $d^c x = dd^c f$ means that $x+df$ is d-closed and d^c-closed, hence $[x]$ belongs to the image of $H^{1,0}_d(M)\oplus\overline{H^{1,0}_d(M)}$. ■

The Lee-Gauduchon space

DEFINITION: Let $W \subset H^{2n-1}(M,\mathbb{R})$ be the space of all cohomology classes α such that $\int_M \alpha\wedge\rho = 0$ for all closed holomorphic forms $\rho \in \Lambda^{1,0}(M,\mathbb{R})$. We call W **the Lee-Gauduchon space**.

REMARK: Since $\int_M d^c u\wedge\rho = \int_M \omega^{n-1}\wedge d^c\rho = 0$, **the image of the natural map $d^c: H_{AE}^{n-1,n-1}(M){\to}H^{2n-1}(M)$ belongs to W.**

PROPOSITION: Let M be a compact complex n-manifold. **Then $W = d^c(H_{AE}^{n-1,n-1}(M,\mathbb{R}))$.**

Proof: Dualizing the exact sequence (*), we obtain

$$H_{AE}^{n-1,n-1}(M,\mathbb{R}) \stackrel{d^c}{\to} H^{2n-1}(M,\mathbb{R}){\to}H^{2n-1}(M,\mathbb{R})/W{\to}0,$$

because W is the annihilator of $H_d^{1,0}\oplus\overline{H_d^{1,0}(M)} \subset H^1(M,\mathbb{R})$. ■

REMARK: **$W=0$ on all manifolds for which the dd^c-lemma holds,** that is, on all projective, Moishezon, Kähler, Fujiki class C etc.

Lee forms and Lee class of a Gauduchon form

DEFINITION: **Lee form** of a Hermitian metric ω is $(d^c)*\omega = \frac{1}{n-1} * d^c(\omega^{n-1})$. If $\omega = f\omega_1$, where ω_1 is Kähler, the Lee form is $d \log f$; in particular, for such ω the Lee form is closed.

REMARK: Clearly, the Lee form is d^*-closed **if and only if ω is Gauduchon.**

DEFINITION: Let ω^{n-1} be a Gauduchon form. The corresponding **Lee-Gauduchon form** is $d^c(\omega^{n-1})$. This form is clearly closed; its **Lee-Gauduchon class** is the class of ω^{n-1} in $H^{2n-1}(M,\mathbb{R})$.

REMARK: Let $W \subset H^{2n-1}(M,\mathbb{R})$ be the space of all cohomology classes α such that $\int_M \alpha \wedge \rho = 0$ for all closed holomorphic forms $\rho \in \Lambda^{1,0}(M)$. Since $\int_M d^c(\omega^{n-1}) \wedge \rho = \int_M \omega^{n-1} \wedge d^c\rho = 0$, **all Lee-Gauduchon classes belong to W.**

The Lee-Gauduchon cone

DEFINITION: The Lee-Gauduchon cone $LG(M) \subset W$ is the set of all Lee-Gauduchon classes of Gauduchon forms.

CLAIM: The Lee-Gauduchon cone **is a convex, open cone in W.**

Proof: The set of Lee-Gauduchon forms is open in W, because it is an image of the Gauduchon cone, which is open in $H_{AE}^{n-1,n-1}(M)$, and $W = d^c(H_{AE}^{n-1,n-1}(M,\mathbb{R}))$. ■

REMARK: $W = 0$ on all manifolds for which the dd^c-lemma holds, that is, on all projective, Moishezon, Kähler, Fujiki class C etc.

DEFINITION: A Bott-Chern class $\alpha \in H_{BC}^{1,1}(M,\mathbb{R})$ is called **exact pseudo-effective** if its image in $H^2(M,\mathbb{R})$ vanishes and it can be represented by a positive, closed (1,1)-current.

Exact pseudoeffective Bott-Chern classes

Theorem 1: Let M be a compact complex manifold, and $\mathcal{C} \subset H^1(M)$ be the set of classes ρ such that $d^c(\rho) \in H^{1,1}_{BC}(M)$ is pseudo-effective. **Then the Lee-Gauduchon cone $LG(M) \subset W \subset H^{2n-1}(M)$ is the dual cone to $\mathcal{C}$.** In other words, $\alpha \in LG(M)$ **iff** $\int_M \alpha \wedge \rho > 0$ **for any** ρ **such that** $d^c\rho \in H^{1,1}_{BC}(M)$**.**

Proof. Step 1: If $\alpha \in LG(M)$, then $\alpha = d^c\omega^{n-1}$ which gives $\int_M \alpha \wedge \rho = \int_M \omega^{n-1} \wedge d^c\rho$. For any non-zero positive current, the integral $\int_M \omega^{n-1} \wedge d^c\rho$ (known as "the mass" of the current) is positive.

Step 2: To prove the converse inclusion, we fix $u \in W$ and apply the Hahn-Banach theorem to the closed affine space

$$V = u + d^c(\text{Aeppli exact } (n-1, n-1)\text{-forms})$$

and the open cone d^c(Gauduchon forms). By Hahn-Banach, these spaces don't intersect if there exists a positive closed 1-current ξ such that $d^c\xi$ is positive, $\int \xi \wedge u = 0$, and $\int \xi \wedge d^c(w) = 0$, for any Aeppli exact $(n-1, n-1)$-form w. The condition $\int_M \xi \wedge d^c(w) = -\int_M d^c\xi \wedge w = 0$ means that $\int_M d^c\xi \wedge w$ for any exact form w, which is equivalent to $d^c w$ being closed. A closed, d^c-exact (1,1)-form is I-invariant, hence also exact. **We obtain that $\alpha \in LG(M)$ if and only if $\langle \alpha, \mathcal{C} \rangle > 0$.** ■

Locally conformally Kähler manifolds

DEFINITION: A complex Hermitian manifold of dimension $\dim_{\mathbb{C}} > 1$ (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form**.

REMARK: This definition **is equivalent to the existence of a Kähler cover $(\tilde M, \tilde\omega) \to M$ such that the deck group Γ acts on $(M, \tilde\omega)$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.** Let $\tilde M$ be a covering such that the pullback $\tilde\theta$ of θ is exact, $df = \tilde\theta$. Then the pullback of $\tilde\omega$ is conformal to a Kähler form $e^{-f}\tilde\omega$.

Lee-Gauduchon cone on LCK manifolds

REMARK: All known compact LCK manifolds **belong to one of three classes: blow-ups of LCK with potential, blow-ups of Oeljeklaus-Toma and Kato.**

THEOREM: (Ornea-V.)
Let (M, ω, θ) be an LCK manifold in any of these classes. **Then $d^c\theta$ is exact pseudoeffective.** In particular, $LG(M) \not\ni 0$ **and M does not admit a balanced metric.**

CONJECTURE:
$d^c\theta$ is exact pseudoeffective on all compact LCK manifolds.

This conjecture was the main motivation for the current work.

Bimeromorphic manifolds

DEFINITION: Let X, Y be complex manifold, and $Z \subset X \times Y$ a closed subvariety such that the projection pof Z to X and Y is proper and generically bijective. Then Z is called **a bimeromorphic map,** and X and Y are called **bimeromorphic.**

THEOREM: (weak factorization theorem)
Any bimeromorphism can be decomposed onto a composition of several blowups and blowdowns with smooth centers.

REMARK: This immediately implies that **bimeromorphic manifolds have the same fundamental group.** Also, **the spaces of global holomorphic forms on bimeromorphic manifolds are naturally isomorphic.**

COROLLARY: Let M_1, M_2 be compact complex n-manifolds which are bimeromorphic. **Then $W(M_1) = W(M_2)$,** where $W \subset H^{2n-1}(M)$ is the subspace defined above.

The Lee-Gauduchon cone is bimeromorphically invariant

THEOREM: Let M_1, M_2 be compact complex n-manifolds which are bimeromorphic. **Then $LG(M_1) = LG(M_2)$.**

Proof. Step 1: Let and θ_1, θ_2 classes in $H^1(M_i)$ which are identified by the natural isomorphism $H^1(M_1) = H^1(M_2)$. We are going to prove that $\theta_1 \in \mathcal{C}(M_1) \Leftrightarrow \theta_2 \in \mathcal{C}(M_2)$.

Step 2: The pushforward of a positive current is always positive; pullback of a current is, in general, not defined. This makes it difficult to identify the pseudoeffective cones of bimeromorphic manifolds. However, the 1-currents θ_i are exact on the universal cover $\tilde{M}_i$: $\tilde{\theta}_i = df_i$, where f_i are generalized functions on $\tilde{M}_i$. If $d^c\theta_1$ is positive, f_1 plurisubharmonic; however, plurisubharmonic function is always L^1_{loc}-integrable, and can be extended over a closed analytic subset. Therefore, pullback **and** the pushforward of a plurisubharmonic function is plurisubharmonic, which implies that f_1 can be lifted to a graph of the bimeromorphic correspondence and pushed forward to a plurisubharmonic function on $\tilde{M}_2$. ■