Lie pencils, hypercomplex geometry and Salamon's theorem

Misha Verbitsky

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Joint work with Yulia Gorginian and Andrey Soldatenkov

Lie pencils

DEFINITION: Let V be a vector space, and $S \subset \operatorname{Hom}(\Lambda^2 V, V)$ a subspace, such that for any $w \in S$, the map $w(x,y)$, denoted in the sequel as $[x,y]_w$, satisfies Jacobi condition $[[x,y]_w,z]_w + [[y,z]_w,x]_w + [[z,x]_w,y]_w = 0$. Then S is called **the Lie pencil**, or **pencil of Lie algebras**.

REMARK: This notion appeared in literature under several names.

- A. B. Yanovski: *"linear bundle of Lie algebras".*
- N. A. Koreshkov: *"Lie sheaves".*
- N. A. Koreshkov: *"Lie pencils".*
- N. A. Koreshkov: *"n-tuple Lie algebras".*
- V. V. Dotsenko and A. S. Khoroshkin: *"algebra with n compatible brackets".*
- I. L. Cantor and D. E. Persits: *"sheaves of linear Poisson brackets".*

Examples of Lie pencils

EXAMPLE: Let $A \in \operatorname{Mat}(R)$ be a matrix. Then the multiplication $X, Y \to XAY$ is associative, hence the bracket $[X,Y]_A := XAY - YAX$ satisfies the Jacobi identity. **This defines a Lie pencil** $\operatorname{Mat}(R) \subset \operatorname{Hom}(\Lambda^2 V, V)$, where $V = \operatorname{Mat}(R)$.

EXAMPLE: Let R be a vector space with scalar product, $A \in \operatorname{Mat}(R)$ be a symmetric matrix, $X, Y \in \mathfrak{so}(R)$ antisymmetric matrices. Then the matrix XAY satisfies $(XAY)^\perp = YAX$, hence $[X,Y]_A := XAY - YAX$ defines a Lie algebra structure on $V = \mathfrak{so}(R)$. **This defines a Lie pencil** $\operatorname{Sym}^2(R) \subset \operatorname{Hom}(\Lambda^2 V, V)$.

EXAMPLE: Let A be an antisymmetric matrix, X, Y symmetric matrices. Then the matrix XAY satisfies $(XAY)^\perp = -YAX$, hence $[X,Y]_A := XAY - YAX$ defines a Lie algebra structure on $V = \operatorname{Sym}^2(R)$. **This defines a Lie pencil** $\mathfrak{so}(R) \subset \operatorname{Hom}(\Lambda^2 V, V)$.

EXAMPLE: Let V be the space of all $m \times n$ matrices and S the space of all $n \times m$ matrices, $A \in S$ and $X, Y \in V$. **Then** $[X,Y]_A := XAY - YAX$ **defines a Lie pencil** $S \subset \operatorname{Hom}(\Lambda^2 V, V)$.

The main conjecture

DEFINITION: A Lie pencil $S \subset \operatorname{Hom}(\Lambda^2 V, V)$ is **S-solvable** if V admits a filtration $V = V_0 \supset V_1 \supset ... \supset V_n = 0$ such that $[V_i, V_i]_w \subset V_{i-1}$ for all $w \in S$, and **S-nilpotent** if V admits a filtration $V = V_0 \supset V_1 \supset ... \supset V_n = 0$ such that $[V_i, V]_w \subset V_{i-1}$ for all $w \in S$.

THE MAIN CONJECTURE: Let $S \subset \operatorname{Hom}(\Lambda^2 V, V)$ be a Lie pencil. Assume that the Lie algebra $(V, [\cdot,\cdot]_w)$ is nilpotent for all $w \in S$. **Then (V, S) is S-solvable.**

REMARK: In these assumptions, (V, S) is not necessarily S-nilpotent. I would describe the counter-example later.

REMARK: We are interested in this conjecture only when $S = \mathbb{H}$ and the Lie pencil comes from a hypercomplex structure on a Lie algebra (more about it later), but it might be true in all generality.

Chevalley-Eilenberg complex

PROPOSITION: Let $w \in \operatorname{Hom}(\Lambda^2 V, V)$. Consider the dual map $d_w : V^* \longrightarrow \Lambda^2 V^*$. Extend this map to $d_w : \Lambda^k V^* \longrightarrow \Lambda^{k+1} V^*$ using the Leibniz rule $d_w(x \wedge y) = d_w(x) \wedge y + (-1)^{\tilde{x}} x \wedge d_w y$. **Then $d_w^2 = 0$ if and only if w defines the Lie algebra structure on V.**

Proof: Left as an exercise. ■

DEFINITION: The complex

$$V^* \xrightarrow{d_w} \Lambda^2 V^* \xrightarrow{d_w} \Lambda^3 V^* \xrightarrow{d_w} \ldots$$

is called **the Chevalley-Eilenberg complex of the Lie algebra $(V, [\cdot,\cdot]_w)$.**

DEFINITION: A map $d \in \operatorname{Hom}(\Lambda^\bullet V^*, \Lambda^{\bullet+1} V^*)$ is called **a differential** if $d^2 = 0$.

Anticommuting differentials and Lie pencils

REMARK: Let $d_1, d\ :_2 \in$ Hom$(\Lambda^\bullet V^*, \Lambda^{\bullet+1} V^*)$ be two differentials. **Then $d_1 + d_2$ is a differential if and only if $d_1, 2_d$ anticommute.**

Proof: $(d_1 + d_2)^2 = d_1^2 + d_2^2 + d_1 d_2 + d_2 d_1 = d_1 d_2 + d_2 d_1$. ■

COROLLARY: Let V be a vector space, and $S \subset$ Hom$(\Lambda^\bullet V^*, \Lambda^{\bullet+1} V^*)$ be a collection of anti-commuting differentials satisfying the Leibniz rule. Then **the dual maps define a Lie pencil $S \to$ Hom$(\Lambda^2 V, V)$.** Moreover, **any Lie pencil is obtained this way.** ■

"Differentials on $\Lambda^\bullet V^*$ define Lie algebra structures on V; anti-commuting differentials define Lie pencils".

REMARK: Anticommuting differentials are pre-eminent in complex geometry: $d, d^c := IdI^{-1}, \partial, \overline{\partial}$ and so on. **This is why we are interested in Lie pencils.**

Complex structures

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: It is sufficient to check the condition $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ on any set of generators of $T^{1,0}M$. In particular, if (M,I) is homogeneous (equipped with a transitive Lie group action preserving I) **it suffices to check it on invariant vector fields**.

Nilmanifolds

DEFINITION: Let M be a smooth manifold equipped with a transitive action of nilpotent Lie group. Then M is called **a nilmanifold**.

REMARK: All nilmanifolds are obtained as quotient spaces, $M = G/H$.

THEOREM: (Malčev)
Let $\mathfrak{g}$ be a nilpotent Lie algebra defined over $\mathbb{Q}$, and G its Lie group. **Then G contains a discrete subgroup Γ such that G/Γ is compact,** and $\Gamma = e^{\Gamma_\mathfrak{g}}$, where $\Gamma_\mathfrak{g}$ is a lattice subalgebra in $\mathfrak{g}$. Moreover, $\mathfrak{g} \cong \Gamma_\mathfrak{g} \otimes_\mathbb{Q} \mathbb{R}$. Finally, **all nilmanifolds are obtained this way.**

REMARK: Topologically, **all simply connected nilpotent Lie groups are diffeomorphic to $\mathbb{R}^n$,** and all nilmanifolds are **iterated circle fibrations**.

Complex nilmanifolds

DEFINITION: **An integrable complex structure** on a real Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: **Any such decomposition defines a complex structure I on $\mathfrak{g}$ by $I\big|_{\mathfrak{g}^{1,0}} = \sqrt{-1}$ and $I\big|_{\mathfrak{g}^{0,1}} = -\sqrt{-1}$.** Integrability of complex structure is given by $[T^{1,0}G, T^{1,0}G] \subset T^{1,0}G$, which is equivalent to $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$.

REMARK: Left-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: Let G be a group equipped with a left-invariant complex structure, and $\Gamma \subset G$ a cocompact lattice. Since Γ acts on G by biholomorphisms, the compact manifold $M = G/\Gamma$ inherits a complex structure. It is called **a complex nilmanifold**.

Iwasawa-type complex structures

EXAMPLE: **Iwasawa manifold** is the quotient of $\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$ (group $N_3(\mathbb{C})$ of upper triangular complex matrices) by a lattice Γ. As an example of Γ we can take $N_3(\mathbb{Z}[\sqrt{-1}\,])$.

REMARK: Let I be a bi-invariant complex structure on a nilpotent Lie group G. Such a complex structure is called **Iwasawa type**.

REMARK: Complex structure on $\mathfrak{g}$ is bi-invariant if and only if $M := (G, I)/\Gamma$ is homogeneous under the left action of G. Then the left invariant vector fields are holomorphic, and this implies that $[X, Y] = 0$ whenever $X \in T^{1,0}M$ and $Y \in T^{0,1}M$.

COROLLARY: Let I be a complex structure on a Lie algebra $\mathfrak{g}$. **Then I is bi-invariant if and only if $[g^{0,1}, \mathfrak{g}^{1,0}] = 0$.**

Proof: If this $[g^{0,1}, \mathfrak{g}^{1,0}] = 0$, then $\mathrm{Lie}_X Y = 0$ is of type (1,0) for any right-invariant vector fields X, Y, with Y of type (1,0). Then $gYg^{-1} \in \mathfrak{g}^{1,0}$ for any $g \in G$, hence $T^{1,0}G$ is bi-invariant. ■

Locally trivial elliptic fibrations

REMARK: **"A surface"** here would always mean **"a compact complex manifold of complex dimension 2".**

REMARK: Let L be a line bundle on a complex manifold X, $\operatorname{Tot} L^*$ the space of non-zero vectors of L, and $\alpha \in \mathbb{C}$ a number satisfying $|\alpha| > 1$, and $M := \operatorname{Tot} L^*/\langle\alpha\rangle$ the quotient by the corresponding $\mathbb{Z}$-action. **Then M is a locally trivial eliptic fibration over X with fiber $\mathbb{C}^*/\langle\alpha\rangle$.**

REMARK: **Any locally trivial elliptic fibration over a curve has this nature.** Its **Chern class** is Chern class of L.

Kodaira surface

DEFINITION: Let T, T' be elliptic curves. **Kodaira surface** $\pi : M \longrightarrow T$ is a locally trivial holomorphic fibration over T with fiber T' and non-trivial Chern class.

A remark on terminology: These are **"primary"** Kodaira surfaces. **"Secondary"** ones are obtained by taking finite unramified quotients.

REMARK: The Kodaira surface is diffeomorphic to a quotient $S^1 \times (G/G_{\mathbb{Z}})$ where G is a 3-dimensional Heisenberg group, and $G_{\mathbb{Z}}$ a lattice in G. Therefore, **Kodaira surface is a nilmanifold**. Its complex structure is left-invariant, but not bi-invariant.

EXERCISE: Check that **the manifold M is a complex nilmanifold, but it is not homogeneous.**

REMARK: **Kodaira surface is not Kähler.** Indeed, the cohomology class of $\pi^*(\omega_T)$ vanishes, where ω_T is the Kähler form on T. **The product of ω_T and the Kahler form on M (if it exists) is a positive volume form,** hence **it cannot be exact.**

Complex nilmanifolds and the Lie pencils

CLAIM: Let (M,I) be an almost complex manifold, and $d^c := IdI^{-1} : \Lambda^k M \longrightarrow \Lambda^{k+1} M$ the twisted de Rham differential. **The almost complex structure I is integrable if and only if d and d^c anticommute**.

Proof: Left as an exercise. ■

COROLLARY: Let $\mathfrak{g}$ be a Lie algebra, and $I \in \operatorname{End}\mathfrak{g}$ an operator which satisfies $I^2 = -\operatorname{Id}$ ("an almost complex structure"). Consider the twisted Lie bracket $[X,Y]_I := I[I^{-1}X, I^{-1}Y]$. **Then I is integrable if and only if the 2-dimensional space S generated by $[\cdot,\cdot]$ and $[\cdot,\cdot]_I$ is a Lie pencil.** ■

DEFINITION: Recall that **the central series** iof a Lie algebra $\mathfrak{g}$ is the sequence $\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset ...$ such that $\mathfrak{g}_i = [\mathfrak{g}, \mathfrak{g}_{i-1}]$.

The Millionschikov's conjecture

QUESTION: (D. Millionschikov)
Let $(\mathfrak{g}, I)$ be a Lie algebra equipped with an integrable complex structure, and N the length of the central series of $\mathfrak{g}$. **Prove that $\frac{N}{\dim_{\mathbb{R}}\mathfrak{g}} \leqslant 2/3$.**

REMARK: Millionschikov discovered a family of algebras $(\mathfrak{g}, I)$ of real dimension $6n$ and with central series of length $4n$, hence **this bound is optimal.**

DEFINITION: Let $S \subset \operatorname{Hom}(\Lambda^2 V, V)$ be a Lie pencil. **An S-ideal** in V is a subspace $V_1 \subset V$ such that $[V, V_1]_w \subset V_1$ for all $w \in S$, and an **S-subalgebra** a subspace $V_1 \subset V$ such that $[V_1, V_1]_w \subset V_1$ for all $w \in S$.

PROPOSITION: Let $\mathfrak{g}$ be the Millionschikov algebra, and $S \subset \operatorname{Hom}(\Lambda^2(\mathfrak{g}), \mathfrak{g})$ be the 2-dimensional Lie pencil associated with the complex structure as above. **Then $(\mathfrak{g}, S)$ is not S-nilpotent**.

Proof: Suppose that $\mathfrak{g}$ is S-nilpotent, and $\mathfrak{g} \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 ...$ the corresponding chain of S-ideals, with $\mathfrak{g}_i$ being generated by $[\mathfrak{g}, \mathfrak{g}_{i-1}]_w$ for all $w \in S$. Using induction, we obtain that $\mathfrak{g}_i$ are I-invariant; indeed, $[\mathfrak{g}, I\mathfrak{g}_i]_I = I[\mathfrak{g}, \mathfrak{g}_i]$. **Then $\dim_{\mathbb{R}} \frac{\mathfrak{g}_i}{\mathfrak{g}_{i+1}} \geqslant 2$, hence $\frac{N}{\dim\mathfrak{g}} \leqslant 1/2$.** However, for Millionshchikov's algebra we have $\frac{N}{\dim_{\mathbb{R}}\mathfrak{g}} = 2/3$, a contradiction. ■

Hypercomplex nilmanifolds

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K: \ TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

DEFINITION: A hypercomplex structure on a Lie algebra $\mathfrak{g}$ is an action of quaternion algebra such that the almost complex structures induced on $\mathfrak{g}$ by I, J, K are integrable.

REMARK: Hypercomplex structures on a Lie algebra are the same as left-invariant hypercomplex structures on the corresponding Lie group.

Hypercomplex nilmanifolds and the Lie pencils

CLAIM: Let $d, d_I := IdI^{-1}, d_J := JdJ^{-1}, d_K := KdK^{-1}$ be the twisted de Rham differentials on a hypercomplex manifold. **Then d, d_I, d_J, d_K anticommute**.

Proof: Clearly, $\{d_I, d_J\} = I\{d, I^{-1}d_J I\}I^{-1}$, and $I^{-1}d_J I = d_K$, hence $\{d_I, d_J\} = I\{d, d_K\}I^{-1}$. This anticommutator vanishes, because K is integrable. ■

REMARK: From this observation we obtain that **a hypercomplex structure on a Lie algebra defines a Lie pencil of dimension 4,** obtained by twisting the Lie bracket with the quaternions.

$\mathbb{H}$-solvable hypercomplex Lie algebras

DEFINITION: Let $(\mathfrak{g}, I, J, K)$ be a complex structure on a nilpotent Lie algebra. For any subspace $\mathfrak{u} \subset \mathfrak{g}$, denote by $\mathbb{H}\mathfrak{u}$ the space $\mathfrak{u} + I\mathfrak{u} + J\mathfrak{u} + K\mathfrak{u}$. Define $\mathfrak{g}_1^{\mathbb{H}} := \mathbb{H}[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}_{i+1}^{\mathbb{H}} := [\mathfrak{g}_i^{\mathbb{H}}, \mathfrak{g}_i^{\mathbb{H}}] + I([\mathfrak{g}_i^{\mathbb{H}}, \mathfrak{g}_i^{\mathbb{H}}])$. The algebra $\mathfrak{g}$ is called **$\mathbb{H}$-solvable** if this sequence terminates.

PROPOSITION: The S-solvability for the Lie pencil S associated to a hypercomplex Lie algebra $\mathfrak{g}$ **is equivalent to $\mathbb{H}$-solvability of this algebra.**

Proof. Step 1: Define $\mathfrak{g}_i^S$ as a subspace of $\mathfrak{g}$ generated by $[\mathfrak{g}_{i-1}, \mathfrak{g}_{i-1}]_w$, for all $w \in S$. Clearly, $\mathfrak{g}$ **is S-solvable if and only if this sequence terminates.** We are going to show that $\mathfrak{g}_i^S = \mathfrak{g}_i^{\mathbb{H}}$ for all i.

Step 2: For any spaces $U, V \subset \mathfrak{g}$ and $L = I, J, K$, we have $L[LU, LV] \supset [U, V]_L$. If U, V are $\mathbb{H}$-invariant, this gives $L[U, V] = [U, V]_L$, hence $\mathbb{H}[U, V] = [U, V] + [U, V]_J + [U, V]_J + [U, V]_K$. **Then $\mathfrak{g}_i^S = \mathfrak{g}_i^{\mathbb{H}}$ implies $\mathfrak{g}_{i+1}^S = \mathfrak{g}_{i+1}^{\mathbb{H}}$.** ■

The "main conjecture" for complex and hypercomplex structures

S-solvability for complex nilmanifolds:
Let $(\mathfrak{g}, I)$ be a complex structure on a nilpotent Lie algebra. Consider the following family of subalgebras, defined inductively: $\mathfrak{g}_1^{\mathbb{C}} = [\mathfrak{g}, \mathfrak{g}] + I([\mathfrak{g}, \mathfrak{g}])$, ..., $\mathfrak{g}_{i+1}^{\mathbb{C}} := [\mathfrak{g}_i^{\mathbb{C}}, \mathfrak{g}_i^{\mathbb{C}}] + I([\mathfrak{g}_i^{\mathbb{C}}, \mathfrak{g}_i^{\mathbb{C}}])$. "The main conjecture" for this particular Lie pencil claims that this sequence terminates. **It was proven by S. Salamon**.

S-solvability for hypercomplex nilmanifolds: "Main conjecture" claims that **any hypercomplex Lie algebra is $\mathbb{H}$-solvable.**

This conjecture is proven only for special cases, but it has many important geometric consequences.

THEOREM: (Yu. Gorginian)
Let $M = G/\Gamma$ be a hypercomplex nilmanifold, and $\mathfrak{g}$ the corresponding Lie algebra. Consider a complex structure of form $L = aI + bJ + cK$, where $a^2 + b^2 + c^2 = 1$. Assume that $\mathfrak{g}$ is $\mathbb{H}$-solvable. **Then the complex manifold (M, L) does not contain complex curves, for all a, b, c outside of a countable set**.

S-solvability for 2-dimensional Lie pencils

We prove the following generalization of Salamon's theorem.

THEOREM: (Gorginian-Soldatenkov-V.)
Let $S \subset \operatorname{Hom}(\Lambda^2 V, V)$ be a 2-dimensional Lie pencil. Assume that for all $w \in S$, the corresponding Lie algebra $(V, [\cdot,\cdot]_w)$ is nilpotent. **Then V is S-solvable.**

To prove it, we translate the notion of Lie pencils to the language of algebraic geometry. The following definition is equivalent to the original definition of Lie pencil.

DEFINITION: A k-dimensional Lie pencil on a vector space $\mathfrak{g}$ is a morphism of vector bundles $\Lambda^2\mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^{k-1}} \longrightarrow \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^{k-1}}(1)$ which satisfies Jacobi identity at each point of the projective space $\mathbb{P}^{k-1}$.