MBM classes on hyperkähler manifolds

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The Kähler cone and its faces

This is **joint work with Ekaterina Amerik**.

DEFINITION: Let M be a compact, Kähler manifold, $\mathsf{Kah} \subset H^{1,1}(M,\mathbb{R})$ is Kähler cone, and $\overline{\mathsf{Kah}}$ its closure in $H^{1,1}(M,\mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\mathsf{Kah}}$ and a hyperplane $V \subset H^{1,1}(M,\mathbb{R})$ which has a non-empry interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)
Let M be a Calabi-Yau manifold. Then the group $\operatorname{Aut}(M)$ of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

REMARK: Today I will describe the Kähler cone on holomorphically symplectic manifolds in terms of topological invariants, called **MBM classes**. These are (roughly speaking) classes of minimal rational curves.

This description was used in our proof of cone conjecture for holomorphically symplectic manifolds.

(-2)-classes on a K3 surface

CLAIM: (Hodge index theorem)
Let M be a Kähler surface. **Then the form $\eta \longrightarrow \int_M \eta\wedge\eta$ has signature $(+,-,-,...)$ on $H^{1,1}(M,\mathbb{R})$.**

DEFINITION: Positive cone $\operatorname{Pos}(M)$ on a Kähler surface is the one of the two components of

$$\{v \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \eta\wedge\eta > 0\}$$

which contains a Kähler form.

DEFINITION: A cohomology class $\eta \in H^2(M,\mathbb{Z})$ on a K3 surface is called **(-2)-class** if $\int_M \eta\wedge\eta = -2$.

REMARK: Let M be a K3 surface, and $\eta \in H^{1,1}(M,\mathbb{Z})$ a (-2)-class. **Then either η or $-\eta$ is effective.** Indeed, $\chi(\eta) = 2 + \frac{\eta^2}{2} = 1$ by Riemann-Roch.

Kähler cone for a K3 surface

THEOREM: Let M be a K3 surface, and S the set of all effective (-2)-classes. **Then** $\operatorname{Kah}(M)$ **is the set of all** $v \in \operatorname{Pos}(M)$ **such that** $\langle v, s\rangle > 0$ **for all** $s \in S$**.**

DEFINITION: A Weyl chamber on a K3 surface is a connected component of $\operatorname{Pos}(M)\backslash S^\perp$, where $S^\perp$ is a union of all planes $s^\perp$ for all (-2)-classes $s \in S$. **The reflection group** of a K3 surface is a group W generated by reflections with respect to all $s \in S$.

REMARK: Clearly, a Weyl chamber is a fundamental domain of W, and W acts transitively on the set of all Weyl chambers. Moreover, **the Kähler cone of M is one of its Weyl chambers.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: **A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right)\left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

DEFINITION: Let (M,I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M,I)$ is called **minimal** if for any $\mathbb{Q}$-effective homology classes $z_1, z_2 \in H_{1,1}(M,I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. A negative rational homology class $z \in H_{1,1}(M,I)$ is called **monodromy birationally minimal** (MBM) if $\gamma(z)$ is minimal and $\mathbb{Q}$-effective for one of birational models (M,I') of (M,I), where $\gamma \in O(H^2(M))$ is an element of the monodromy group of (M,I).

This property is **deformationally invariant**.

THEOREM: Let $z \in H^2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type (1,1) with respect to I and I'. Then **η is MBM in (M,I) $\Leftrightarrow$ it is MBM in (M,I')**.

MBM classes for $\operatorname{Pic}(M) = \mathbb{Z}$

The MBM and divisorial classes are better understood if the Picard group has rank one and generated by a negative vector (in this case M is non-algebraic).

THEOREM: Let (M, I) be a hyperkähler manifold, $\operatorname{rk}\operatorname{Pic}(M, I) = 1$, and $z \in H_{1,1}(M, I)$ a non-zero negative class. **Then z is monodromy birationally minimal if and only if $\pm z$ is $\mathbb{Q}$-effective.**

Proof. Step 1: If (M, I) has a rational curve, it is by definition minimal, hence represents an MBM class.

Step 2: If (M, I) has no rational curves, it has no exceptional divisors, hence **its Kähler cone is equal to the positive cone** (Huybrechts, Boucksom). Therefore, z is orthogonal to a Kähler class, and hence non-effective. ■

REMARK: This argument proves that **MBM classes correspond to faces of a Kähler cone** for $\operatorname{rk}\operatorname{Pic}(M, I) = 1$.

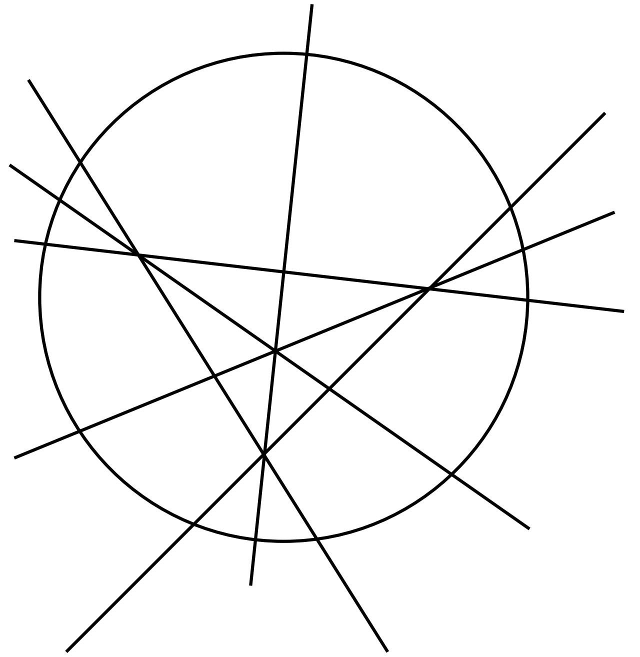
MBM classes and the Kähler cone

THEOREM: Let (M,I) be a hyperkähler manifold, and $S \subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \mid \ z \in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I) \backslash \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

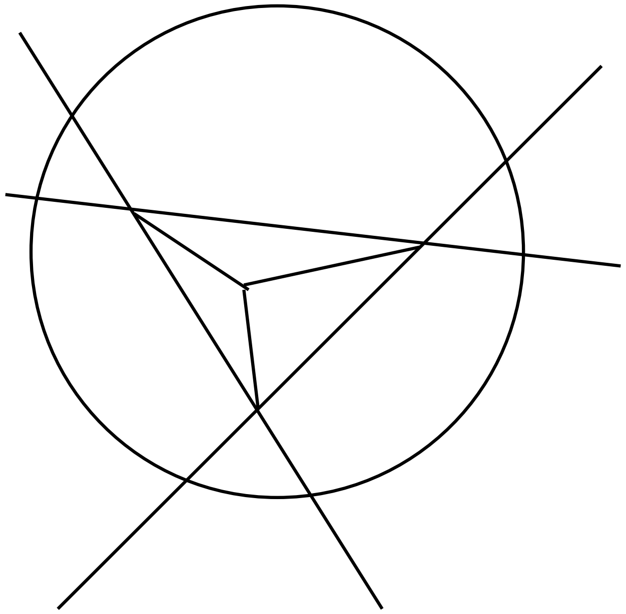
REMARK: This implies that **MBM classes correspond to faces of the Kähler cone.**

MBM classes and the Kähler cone: the picture

REMARK: This implies that $z^\perp \cap \mathsf{Pos}(M,I)$ **either has dense intersection with the interior of the Kähler chambers (if z is not MBM), or is a union of walls of those (if z is MBM);** that is, there are no "barycentric partitions" in the decomposition of the positive cone into the Kähler chambers.



Allowed partition



Prohibited partition

Families of rational curves: lower bound on dimension

THEOREM: (Z. Ran)
Let M be a hyperkähler manifold of dimension $2n$. **Then any rational curve $C \subset M$ deforms in a family of dimension at least $2n-2$.**

Proof: By adjunction formula, $\deg(NC) = -2$ and $\operatorname{rk}(NC) = 2n-1$, which implies that C deforms in a family of dimension at least $2n-3$. The extra parameter is due to the existence of the twistor space $\operatorname{Tw}(M)$. This is a complex manifold of dimension $n+1$, fibered over $\mathbb{C}P^1$ in such a way that M is one of the fibers and the other fibers correspond to the other complex structures coming from the hyperkähler action on M. The same adjunction argument shows that C deforms in $\operatorname{Tw}(M)$ in a family of dimension at least $2n-2$. But **all deformations of C are contained in M since the neighbouring fibers contain no curves.** ■

Families of rational curves: coisotropicity

DEFINITION: A complex analytic subvariety Z of a holomorphically symplectic manifold (M, Ω) is called **isotropic** if $\Omega|_Z = 0$ and **coisotropic** if Ω has rank $\frac{1}{2}\dim_{\mathbb{C}} M - \operatorname{codim}_{\mathbb{C}} Z$ on TZ in all smooth points of Z, which is the minimal possible rank for a $2n-p$-dimensional subspace in a $2n$-dimensional symplectic space.

THEOREM: Let M be a hyperkähler manifold, $C \subset M$ a minimal rational curve, and $Z \subset M$ the union of all deformations of C in M. **Then Z is a coisotropic subvariety of M.**

Proof. Step 1: Let V be a MRC quotient of Z. Since fibers of $\pi: Z \longrightarrow V$ are rationally connected, they are isotropic.

Step 2: Let $k := \operatorname{codim} Z$. Let T be the irreducible component of the parameter space for deformations of C in M. We have $dim(T) \geqslant 2n-2$ by Ziv Ran. Therefore the dimension of the universal family of curves over T is at least $2n-1$. Since it projects onto Z which is $2n-k$-dimensional, the fibers of this projection are of dimension at least $k-1$.

Families of rational curves: coisotropicity (2)

THEOREM: Let M be a hyperkähler manifold, $C \subset M$ a minimal rational curve, and $Z \subset M$ the union of all deformations of C in M. **Then Z is a coisotropic subvariety of M.**

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Step 3: By bend-and-break, **there is only a finite number of minimal rational curves through two general points.** This means that the fibers of the MRC fibration $\pi : Z \longrightarrow V$ are at least k-dimensional, and $\dim V \leqslant \dim M - 2k$.

Step 4: Since the fibers of π are isotropic, one has $\operatorname{rk} \Omega|_Z \leqslant \dim V = 1/2 \dim M - k$, hence Z is coisotropic, and the inequality $\dim V \leqslant \dim M - 2k$ is equality. ■

Families of rational curves: upper bound

COROLLARY: The deformation space of minimal rational curves on a holomorphic symplectic manifold is $2n-2$-dimensional.

Proof. Step 1: Let C be a minimal rational curve, and Z the union of all its deformations $k := \operatorname{codim} Z$. Consider the MRC map $\pi: Z \longrightarrow V$. We have shown that $\dim V = \dim M - 2k$.

Step 2: Since $\dim V = \dim M - 2k$, the fibers of $\pi: Z \longrightarrow V$ are k-dimensional. Applying bend-and-break again, we obtain that there is a $2k-2$-dimensional family of deformations of C in each fiber of π. ■

Families of rational curves: deformational invariance (local)

COROLLARY: Let C be a minimal rational curve in a hyperkähler manifold M_0. **Then any small deformation M_t of $M = M_0$ such that the homology class z of C stays of type $(1,1)$ on M_t, contains a deformation of C.**

Proof: From Riemann-Roch theorem it follows that C deforms in a family of dimension at least $2n-3+\dim(\operatorname{Def}(M))$. Since the deformations of C inside any M_t form a family of dimension $2n-2$ (when nonempty), the conclusion follows. ■

Families of rational curves: deformational invariance (global)

COROLLARY: If C is minimal, **any deformation M_t of $M = M_0$ such that the corresponding homology class remains of type $(1,1)$ has a birational model containing a rational curve in that homology class.**

Proof: Let $\operatorname{Teich}(M)^0$ be the connected component of the Teichmüller space of M containing the parameter point for our complex manifold M_0, and $\operatorname{Teich}_z(M)^0$ the part of it where z remains of type (1,1). Connecting M_t with M_0 by a path and applying the above corollary, we obtain the proof.

Birational models appear since $\operatorname{Teich}_z(M)$ is not Hausdorff, so that at the end of a path we might arrive to another point of $\operatorname{Teich}_z(M)$, not separable from M_t. However, a theorem of Huybrechts implies that unseparable points of $\operatorname{Teich}_z(M)$ correspond to birational complex manifolds. ■

REMARK: This proves the deformational invariance of MBM classes.