Towards the cone conjecture for hyperkähler manifolds

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The Kähler cone and its faces

The work presented here (on the slides 13-15) **is done in collaboration with Ekaterina Amerik.**

DEFINITION: Let M be a compact, Kähler manifold, $\operatorname{Kah} \subset H^{1,1}(M, \mathbb{R})$ is Kähler cone, and $\overline{\operatorname{Kah}}$ its closure in $H^{1,1}(M, \mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\operatorname{Kah}}$ and a hyperplane $V \subset H^{1,1}(M, \mathbb{R})$ which has a non-empry interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)
Let M be a Calabi-Yau manifold. Then the group $\operatorname{Aut}(M)$ of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

Birational Kähler cone

REMARK: Define **pseudo-isomorphism** $M \longrightarrow M'$ as a birational map which is an isomorphism outside of codimension $\geqslant 2$ subsets of M, M'.

REMARK: For any pseudo-isomorphic manifolds M, M', one has $H^2(M) = H^2(M')$.

DEFINITION: Movable Kähler cone, also known as **birational Kähler cone** and **birational nef cone** is a closure of a union of $\mathrm{Kah}(M')$ for all M' pseudo-isomorphic to M.

CONJECTURE: (Morrison-Kawamata birational cone conjecture)
Let M be a Calabi-Yau manifold. Then the group $\mathrm{Bir}(M)$ if birational automorphisms of M **acts on the set of faces of the movable cone with finite number of orbits.**

(-2)-classes on a K3 surface

CLAIM: (Hodge index theorem)
Let M be a Kähler surface. **Then the form $\eta \longrightarrow \int_M \eta\wedge\eta$ has signature $(+,-,-,...)$ on $H^{1,1}(M,\mathbb{R})$.**

DEFINITION: Positive cone $\operatorname{Pos}(M)$ on a Kähler surface is the one of the two components of

$$\{v \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ \int_M \eta\wedge\eta > 0\}$$

which contains a Kähler form.

DEFINITION: A cohomology class $\eta\in H^2(M,\mathbb{Z})$ on a K3 surface is called **(-2)-class** if $\int_M \eta\wedge\eta = -2$.

REMARK: Let M be a K3 surface, and $\eta\in H^{1,1}(M,\mathbb{Z})$ a (-2)-class. **Then either η or $-\eta$ is effective.** Indeed, $\chi(\eta) = 2+\frac{\eta^2}{2} = 1$ by Riemann-Roch.

Kähler cone for a K3 surface

THEOREM: Let M be a K3 surface, and S the set of all effective (-2)-classes. **Then $\operatorname{Kah}(M)$ is the set of all $v \in \operatorname{Pos}(M)$ such that $\langle v, s\rangle > 0$ for all $s \in S$.**

DEFINITION: A Weyl chamber on a K3 surface is a connected component of $\operatorname{Pos}(M)\backslash S^\perp$, where $S^\perp$ is a union of all planes $s^\perp$ for all (-2)-classes $s \in S$. **The reflection group** of a K3 surface is a group W generated by reflections with respect to all $s \in S$.

REMARK: Clearly, a Weyl chamber is a fundamental domain of W, and W acts transitively on the set of all Weyl chambers. Moreover, **the Kähler cone of M is one of its Weyl chambers.**

Cone conjecture for a K3 surface

THEOREM: Let M be a K3 surface. **Then $\operatorname{Aut}(M)$ is the group of all isometries of $H^{1,1}(M,\mathbb{Z})$ preserving the Kähler chamber.**

COROLLARY: Morrison-Kawamata cone conjecture holds for a K3 surface.

Proof. Step 1: A group Γ of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \ | \ l^2 = x\}$ for any given x (see Kneser, *Quadratische Formen*, Satz 30.2). **Therefore, Γ acts with finitely many orbits on the set of (-2)-vectors in Λ.**

Step 2: For each pair of faces F, F' of a Kähler cone and $w \in O(\Lambda)$ mapping F to F', w maps Kah to itself or to an adjoint Weyl chamber K'. Then $K' = r(K)$, where r is the reflection fixing F'. In the first case, $w \in \operatorname{Aut}(M)$. In the second case, rw maps F to F' and maps Kah to itself, hence $rw \in \operatorname{Aut}(M)$. ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right)\left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Birational nef cone on a hyperkähler manifold

DEFINITION: A cohomology class $\nu \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive, closed current.

THEOREM: (Huybrechts, Boucksom)
On any hyperkähler manifold M, **birational Kähler cone is dual (with respect to the BBF pairing) to the pseudoeffective cone.** Moreover, the birational Kähler cone is a union of Kähler cones **for all hyperkähler manifolds M' pseudo-isomorphic to M.**

THEOREM: Divisorial Zariski decomposition (Boucksom).
For any pseudoeffective ν, we have $\nu = \nu_0 + \sum \alpha_i E_i$, where ν_0 is birationally nef, α_i positive numbers, and E_i are exceptional divisors.

COROLLARY: Let $\eta \in \operatorname{Pos}(M)$ be an element of a positive cone on a hyperkähler manifold. **Then η is birationally nef if and only if $q(\eta, E) \geqslant 0$ for any exceptional divisor E.**

REMARK: In other words, **the faces of birational Kähler cone are dual to the classes of exceptional divisors.**

Divisorial reflections and Weyl chambers

DEFINITION: Monodromy group Γ of a hyperkähler manifold is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ generated by monodromy operators for all Gauss-Manin local system associated with complex deformations of M.

REMARK: The **monodromy group of a hyperkähler manifold is a finite index subgroup in** $O(H^2(M,\mathbb{Z}),q)$ (follows from global Torelli).

THEOREM: (Markman)
For each exceptional divisor E on a hyperkähler manifold, **there exists a reflection** $r_E \in O(H^2(M,\mathbb{Z}))$ **in the monodromy group fixing** $E^\perp$**.**

DEFINITION: Such a reflection is called **a divisorial reflection**.

DEFINITION: Weyl chamber on a hyperkähler manifold is a connected component of $\operatorname{Pos}(M)\backslash E^\perp$, where $E^\perp$ is a union of all planes $e^\perp$ for all exceptional divisors e.

REMARK: A Weyl chamber is a fundamental domain of a group generated by divisorial reflections. **Birational Kähler cone is one of the Weyl chambers.**

The proof of birational cone conjecture

THEOREM: Let M be a hyperkähler manifold. **Then** $\mathrm{Bir}(M)$ **is the group of all** $\gamma \in \Gamma$ **preserving the birational Kähler chamber** Kah_B**.** Here Γ is the monodromy group.
Proof: Follows from global Torelli. ∎

COROLLARY: (Markman) Birational Morrison-Kawamata cone conjecture holds for hyperkähler manifolds.

Proof. Step 1: Let δ be the discriminant of a lattice $H^2(M,\mathbb{Z})$, and E an exceptional divisor. **Then** $|E^2| \leqslant 2\delta$**.** Indeed, otherwise the reflection $x \longrightarrow x - 2\frac{q(x,E)}{q(E,E)}E$ would not be integer.

Step 2: A group of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \mid l^2 = x\}$ for any given x (see Kneser, *Quadratische Formen*, Satz 30.2). Therefore, **Γ acts with finitely many orbits on the set of classes of exceptional divisors.**

Step 3: For each pair of faces F, F' of a birational Kähler cone and $w \in O(\Lambda)$ mapping F to F', w maps Kah_B to itself or to an adjoint Weyl chamber K'. Then $K' = r(K)$, where r is the reflection fixing F'. In the first case, $w \in \mathrm{Aut}(M)$. In the second case, rw maps F to F' and maps Kah_B to itself, hence $rw \in \mathrm{Aut}(M)$. ∎

MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M,\mathbb{R})$ satisfying $q(\eta,\eta) < 0$.

DEFINITION: Let (M,I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M,I)$ is called **minimal** if for any $\mathbb{Q}$-effective homology classes $z_1, z_2 \in H_{1,1}(M,I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. A negative rational homology class $z \in H_{1,1}(M,I)$ is called **monodromy birationally minimal** (MBM) if $\gamma(z)$ is minimal and $\mathbb{Q}$-effective for one of birational models (M,I') of (M,I), where $\gamma \in O(H^2(M))$ is an element of the monodromy group of (M,I).

DEFINITION: Let (M,I) be a hyperkaehler manifold. A negative rational class $z \in H^{1,1}_{\mathbb{Q}}(M,I)$ is called **divisorial** if $z = \lambda[D]$ for some divisor D and $\lambda \in \mathbb{Q}$.

These properties are **deformationally invariant**.

THEOREM: Let $z \in H^2(M,\mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type (1,1) with respect to I and I'. Then

* **η is divisorial in $(M,I) \Leftrightarrow$ it is divisorial in (M,I').**
* **η is MBM in $(M,I) \Leftrightarrow$ it is MBM in (M,I').**

MBM classes for $\operatorname{Pic}(M) = \mathbb{Z}$

The MBM and divisorial classes are better understood if the Picard group has rank one and generated by a negative vector (in this case M is non-algebraic).

THEOREM: Let (M, I) be a hyperkähler manifold, $\operatorname{rk}\operatorname{Pic}(M, I) = 1$, and $z \in H_{1,1}(M, I)$ a non-zero negative class. **Then z is monodromy birationally minimal if and only if $\pm z$ is $\mathbb{Q}$-effective.**

Proof. Step 1: If (M, I) has a rational curve, it is by definition minimal, hence represents an MBM class.

Step 2: If (M, I) has no rational curves, it has no exceptional divisors, hence **birational Kähler cone is equal to the positive cone** (Boucksom).

Step 3: If (M, I) has no rational curves, any pseudo-isomorphism from (M, I) to another hyperkähler manifold must be trivial. Indeed, pseudo-isomorphisms are birational, and exceptional locus of a birational map is covered by rational curves. Then $\operatorname{Kah}_B = \operatorname{Kah}$: **birational Kähler cone is Kähler cone is positive cone.** ■

REMARK: This argument proves that **MBM classes correspond to faces of a Kähler cone** for $\operatorname{rk}\operatorname{Pic}(M, I) = 1$.

MBM classes and the Kähler cone

THEOREM: Let (M,I) be a hyperkähler manifold, and $S\subset H_{1,1}(M,I)$ the set of all MBM classes in $H_{1,1}(M,I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \ \mid \ z\in S\}$ in $H^{1,1}(M,I)$. **Then the Kähler cone of (M,I) is a connected component of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$**, where $\operatorname{Pos}(M,I)$ is a positive cone of (M,I). Moreover, for any connected component K of $\operatorname{Pos}(M,I)\backslash \cup S^\perp$, there exists $\gamma\in O(H^2(M))$ in a monodromy group of M, and a hyperkähler manifold (M,I') birationally equivalent to (M,I), such that $\gamma(K)$ is a Kähler cone of (M,I').

REMARK: MBM classes correspond to faces of the Kähler cone.

REMARK: Morrison-Kawamata cone conjecture would follow if we prove that monodromy group acts on the set of MBM rays with finitely many orbits. This is implied by the following conjecture.

CONJECTURE: Let M be a hyperkähler manifold. **Then there exists a constant $C>0$, such that for any minimal rational curve $S\subset (M,I)$ with $q(S,S)<0$, one has $|q(S,S)|<C$.**