

Towards the cone conjecture for hyperkähler manifolds

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The Kähler cone and its faces

The work presented here (on the slides 13-15) **is done in collaboration with Ekaterina Amerik.**

DEFINITION: Let M be a compact, Kähler manifold, $\text{Kah} \subset H^{1,1}(M, \mathbb{R})$ is Kähler cone, and $\overline{\text{Kah}}$ its closure in $H^{1,1}(M, \mathbb{R})$, called **the nef cone**. A **face** of a Kähler cone is an intersection of the boundary of $\overline{\text{Kah}}$ and a hyperplane $V \subset H^{1,1}(M, \mathbb{R})$ which has a non-empty interior.

CONJECTURE: (Morrison-Kawamata cone conjecture)

Let M be a Calabi-Yau manifold. Then the group $\text{Aut}(M)$ of biholomorphic automorphisms of M acts on the set of faces of Kah **with finite number of orbits.**

Birational Kähler cone

REMARK: Define **pseudo-isomorphism** $M \rightarrow M'$ as a birational map which is an isomorphism outside of codimension ≥ 2 subsets of M, M' .

REMARK: For any pseudo-isomorphic manifolds M, M' , one has $H^2(M) = H^2(M')$.

DEFINITION: **Movable Kähler cone**, also known as **birational Kähler cone** and **birational nef cone** is a closure of a union of $\text{Kah}(M')$ for all M' pseudo-isomorphic to M .

CONJECTURE: (Morrison-Kawamata birational cone conjecture)

Let M be a Calabi-Yau manifold. Then the group $\text{Bir}(M)$ of birational automorphisms of M **acts on the set of faces of the movable cone with finite number of orbits.**

(−2)-classes on a K3 surface**CLAIM: (Hodge index theorem)**

Let M be a Kähler surface. **Then the form $\eta \mapsto \int_M \eta \wedge \eta$ has signature $(+, -, -, \dots)$ on $H^{1,1}(M, \mathbb{R})$.**

DEFINITION: Positive cone $\text{Pos}(M)$ on a Kähler surface is the one of the two components of

$$\{v \in H^{1,1}(M, \mathbb{R}) \mid \int_M v \wedge v > 0\}$$

which contains a Kähler form.

DEFINITION: A cohomology class $\eta \in H^2(M, \mathbb{Z})$ on a K3 surface is called **(−2)-class** if $\int_M \eta \wedge \eta = -2$.

REMARK: Let M be a K3 surface, and $\eta \in H^{1,1}(M, \mathbb{Z})$ a (−2)-class. **Then either η or $-\eta$ is effective.** Indeed, $\chi(\eta) = 2 + \frac{\eta^2}{2} = 1$ by Riemann-Roch.

Kähler cone for a K3 surface

THEOREM: Let M be a K3 surface, and S the set of all effective (-2) -classes. **Then $\text{Kah}(M)$ is the set of all $v \in \text{Pos}(M)$ such that $\langle v, s \rangle > 0$ for all $s \in S$.**

DEFINITION: A **Weyl chamber** on a K3 surface is a connected component of $\text{Pos}(M) \setminus S^\perp$, where S^\perp is a union of all planes s^\perp for all (-2) -classes $s \in S$. **The reflection group** of a K3 surface is a group W generated by reflections with respect to all $s \in S$.

REMARK: Clearly, a Weyl chamber is a fundamental domain of W , and W acts transitively on the set of all Weyl chambers. Moreover, **the Kähler cone of M is one of its Weyl chambers.**

Cone conjecture for a K3 surface

THEOREM: Let M be a K3 surface. **Then $\text{Aut}(M)$ is the group of all isometries of $H^{1,1}(M, \mathbb{Z})$ preserving the Kähler chamber.**

COROLLARY: **Morrison-Kawamata cone conjecture holds for a K3 surface.**

Proof. Step 1: A group Γ of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \mid l^2 = x\}$ for any given x (see Kneser, *Quadratische Formen*, Satz 30.2). **Therefore, Γ acts with finitely many orbits on the set of (-2) -vectors in Λ .**

Step 2: For each pair of faces F, F' of a Kähler cone and $w \in O(\Lambda)$ mapping F to F' , w maps Kah to itself or to an adjoint Weyl chamber K' . Then $K' = r(K)$, where r is the reflection fixing F' . In the first case, $w \in \text{Aut}(M)$. In the second case, rw maps F to F' and maps Kah to itself, hence $rw \in \text{Aut}(M)$. ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K , satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K .

REMARK: A hyperkähler manifold **has three symplectic forms**

$$\omega_I := g(I\cdot, \cdot), \quad \omega_J := g(J\cdot, \cdot), \quad \omega_K := g(K\cdot, \cdot).$$

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K .

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M .

REMARK: A hyperkähler manifold can be defined as a manifold which **has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2, 0)$ -form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on (M, I) .

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = c q(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$, and $c > 0$ a rational number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki's relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$, where ω is a Kähler form.

Birational nef cone on a hyperkähler manifold

DEFINITION: A cohomology class $\nu \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive, closed current.

THEOREM: (Huybrechts, Boucksom)

On any hyperkähler manifold M , **birational Kähler cone is dual (with respect to the BBF pairing) to the pseudoeffective cone.** Moreover, the birational Kähler cone is a union of Kähler cones **for all hyperkähler manifolds M' pseudo-isomorphic to M .**

THEOREM: Divisorial Zariski decomposition (Boucksom).

For any pseudoeffective ν , we have $\nu = \nu_0 + \sum \alpha_i E_i$, where ν_0 is birationally nef, α_i positive numbers, and E_i are exceptional divisors.

COROLLARY: Let $\eta \in \text{Pos}(M)$ be an element of a positive cone on a hyperkähler manifold. **Then η is birationally nef if and only if $q(\eta, E) \geq 0$ for any exceptional divisor E .**

REMARK: In other words, **the faces of birational Kähler cone are dual to the classes of exceptional divisors.**

Divisorial reflections and Weyl chambers

DEFINITION: Monodromy group Γ of a hyperkähler manifold is a subgroup of $O(H^2(M, \mathbb{Z}), q)$ generated by monodromy operators for all Gauss-Manin local system associated with complex deformations of M .

REMARK: The **monodromy group of a hyperkähler manifold is a finite index subgroup in $O(H^2(M, \mathbb{Z}), q)$** (follows from global Torelli).

THEOREM: (Markman)

For each exceptional divisor E on a hyperkähler manifold, **there exists a reflection $r_E \in O(H^2(M, \mathbb{Z}))$ in the monodromy group fixing E^\perp .**

DEFINITION: Such a reflection is called **a divisorial reflection**.

DEFINITION: Weyl chamber on a hyperkähler manifold is a connected component of $\text{Pos}(M) \setminus E^\perp$, where E^\perp is a union of all planes e^\perp for all exceptional divisors e .

REMARK: A Weyl chamber is a fundamental domain of a group generated by divisorial reflections. **Birational Kähler cone is one of the Weyl chambers.**

The proof of birational cone conjecture

THEOREM: Let M be a hyperkähler manifold. **Then $\text{Bir}(M)$ is the group of all $\gamma \in \Gamma$ preserving the birational Kähler chamber Kah_B .** Here Γ is the monodromy group.

Proof: Follows from global Torelli. ■

COROLLARY: (Markman) Birational Morrison-Kawamata cone conjecture holds for hyperkähler manifolds.

Proof. Step 1: Let δ be the discriminant of a lattice $H^2(M, \mathbb{Z})$, and E an exceptional divisor. **Then $|E^2| \leq 2\delta$.** Indeed, otherwise the reflection $x \longrightarrow x - 2\frac{q(x, E)}{q(E, E)}E$ would not be integer.

Step 2: A group of isometries of a lattice Λ acts with finitely many orbits on the set $\{l \in \Lambda \mid l^2 = x\}$ for any given x (see Kneser, *Quadratische Formen*, Satz 30.2). Therefore, **Γ acts with finitely many orbits on the set of classes of exceptional divisors.**

Step 3: For each pair of faces F, F' of a birational Kähler cone and $w \in O(\Lambda)$ mapping F to F' , w maps Kah_B to itself or to an adjoint Weyl chamber K' . Then $K' = r(K)$, where r is the reflection fixing F' . In the first case, $w \in \text{Aut}(M)$. In the second case, rw maps F to F' and maps Kah_B to itself, hence $rw \in \text{Aut}(M)$. ■

MBM classes

DEFINITION: Negative class on a hyperkähler manifold is $\eta \in H^2(M, \mathbb{R})$ satisfying $q(\eta, \eta) < 0$.

DEFINITION: Let (M, I) be a hyperkähler manifold. A rational homology class $z \in H_{1,1}(M, I)$ is called **minimal** if for any \mathbb{Q} -effective homology classes $z_1, z_2 \in H_{1,1}(M, I)$ satisfying $z_1 + z_2 = z$, the classes z_1, z_2 are proportional. A negative rational homology class $z \in H_{1,1}(M, I)$ is called **monodromy bi-rationally minimal** (MBM) if $\gamma(z)$ is minimal and \mathbb{Q} -effective for one of birational models (M, I') of (M, I) , where $\gamma \in O(H^2(M))$ is an element of the monodromy group of (M, I) .

DEFINITION: Let (M, I) be a hyperkaehler manifold. A negative rational class $z \in H_{\mathbb{Q}}^{1,1}(M, I)$ is called **divisorial** if $z = \lambda[D]$ for some divisor D and $\lambda \in \mathbb{Q}$.

These properties are **deformationally invariant**.

THEOREM: Let $z \in H^2(M, \mathbb{Z})$ be negative, and I, I' complex structures in the same deformation class, such that η is of type $(1,1)$ with respect to I and I' . Then

- * η is divisorial in $(M, I) \Leftrightarrow$ it is divisorial in (M, I') .
- * η is MBM in $(M, I) \Leftrightarrow$ it is MBM in (M, I') .

MBM classes for $\text{Pic}(M) = \mathbb{Z}$

The MBM and divisorial classes are better understood if the Picard group has rank one and generated by a negative vector (in this case M is non-algebraic).

THEOREM: Let (M, I) be a hyperkähler manifold, $\text{rk Pic}(M, I) = 1$, and $z \in H_{1,1}(M, I)$ a non-zero negative class. **Then z is monodromy birationally minimal if and only if $\pm z$ is \mathbb{Q} -effective.**

Proof. Step 1: If (M, I) has a rational curve, it is by definition minimal, hence represents an MBM class.

Step 2: If (M, I) has no rational curves, it has no exceptional divisors, hence **birational Kähler cone is equal to the positive cone** (Boucksom).

Step 3: If (M, I) has no rational curves, any pseudo-isomorphism from (M, I) to another hyperkähler manifold must be trivial. Indeed, pseudo-isomorphisms are birational, and exceptional locus of a birational map is covered by rational curves. Then $\text{Kah}_B = \text{Kah}$: **birational Kähler cone is Kähler cone is positive cone.** ■

REMARK: This argument proves that **MBM classes correspond to faces of a Kähler cone** for $\text{rk Pic}(M, I) = 1$.

MBM classes and the Kähler cone

THEOREM: Let (M, I) be a hyperkähler manifold, and $S \subset H_{1,1}(M, I)$ the set of all MBM classes in $H_{1,1}(M, I)$. Consider the corresponding set of hyperplanes $S^\perp := \{W = z^\perp \mid z \in S\}$ in $H^{1,1}(M, I)$. **Then the Kähler cone of (M, I) is a connected component of $\text{Pos}(M, I) \setminus \cup S^\perp$** , where $\text{Pos}(M, I)$ is a positive cone of (M, I) . Moreover, for any connected component K of $\text{Pos}(M, I) \setminus \cup S^\perp$, there exists $\gamma \in O(H^2(M))$ in a monodromy group of M , and a hyperkähler manifold (M, I') birationally equivalent to (M, I) , such that $\gamma(K)$ is a Kähler cone of (M, I') .

REMARK: MBM classes correspond to faces of the Kähler cone.

REMARK: Morrison-Kawamata cone conjecture would follow if we prove that monodromy group acts on the set of MBM rays with finitely many orbits. This is implied by the following conjecture.

CONJECTURE: Let M be a hyperkähler manifold. **Then there exists a constant $C > 0$, such that for any minimal rational curve $S \subset (M, I)$ with $q(S, S) < 0$, one has $|q(S, S)| < C$.**