Morse-Novikov cohomology and Kodaira-type embedding theorem for locally conformally Kähler manifolds

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Plan.

1. Motivation: Kodaira theorem on embedding of Kähler manifolds into $\mathbb{C}P^n$. Discussion of a posible non-Kähler analogue.

2. LCK manifolds: Locally conformally Kähler (LCK) manifolds, and their definition.

3. Deformational stability problem for LCK manifolds: Counterexamples to deformational stability. Vaisman manifolds and Inoue manifolds. A new notion: **LCK manifolds with potential.**

4. Hopf manifolds. The "LCK with potential" embedding theorem - an LCK analogue of Kodaira's theorem.

5. Cohomology theories for LCK manifolds. Morse-Novikov class, Bott-Chern class, and the applications to embeddiong.

6. Some open questions: A locally conformally Kähler dd^c-lemma and its applications.

Kodaira's theorem

DEFINITION: A **Kähler manifold** is a complex, Hermitian manifold, with a Hermitian form $\omega \in \Lambda^{1,1}(M)$ which is closed: $d\omega = 0$.

REMARK: A complex submanifold of a Kähler manifold is again Kähler. Indeed, a restriction of a closed form is closed.

THEOREM: Let M be a compact, Kähler manifold, ω its Kähler form, $[\omega] \in H^2(M,\mathbb{R})$ its cohomology class. Assume that $[\omega]$ is rational: $[\omega] \in H^2(M,\mathbb{Q})$. **Then M admits a complex embedding into $\mathbb{C}P^n$.**

QUESTION: Can we obtain a similar result for other classes of (non-Kähler) manifolds?

QUESTION: More specifically: find a class of geometric structures which are **inherited by complex subvarieties,** and any complex manifold with such a structure **admits a complex embedding to a model space**.

Locally conformally Kähler (LCK) manifolds

REMARK: It is **always assumed** that LCK manifolds have $\dim_{\mathbb{C}} > 1$.

DEFINITION: Let M be a complex manifold, $\tilde{M} \longrightarrow M$ its covering, $\tilde{M}/\Gamma = M$, and Γ the **monodromy group** freely acting on $\tilde{M}$. Assume that $\tilde{M}$ is Kähler, and Γ acts on $\tilde{M}$ by homotheties. Then M is called **locally conformally Kähler (LCK).**

EXAMPLE: A classical Hopf manifold $\mathbb{C}^n \backslash 0/x \sim qx$ is obviously LCK, if $|q| > 1$.

REMARK: A Hopf manifold is non-Kähler. Indeed, **it is diffeomorphic to $S^1 \times S^{2n-1}$.**

REMARK: All non-Kähler compact complex surfaces are LCK, except one of three classes of Inoue surfaces, and those class VII surfaces which have no spherical shells (by Kato's conjecture, all class VII surfaces have spherical shells).

REMARK: A non-Kähler simply connected manifold can never be LCK. Hence e.g. **Calabi-Eckmann manifolds are not LCK.** Also, the **non-Kähler twistor spaces of compact 4-manifolds are never LCK** (Ornea-V.-Vuletescu).

LCK manifolds and Hermitian geometry

CLAIM: If (M, I, ω) is a Hermitian manifold with $d\omega = \theta \wedge \omega$, for some closed 1-form θ, **then (M, I) is LCK.**

Proof: For some covering of M the pullback $\tilde{\theta}$ of θ is exact, $\tilde{\theta} = d\varphi$, and there **the pullback of ω is conformal to a Kähler form $\tilde{\omega} = e^{-\varphi}\omega$.** ■

REMARK: The converse is also true. Given an LCK manifold M with a Kähler covering $(\tilde{M} \xrightarrow{\pi} M, \tilde{\omega})$, consider a Hermitian form ω on M such that $\pi^*\omega$ is conformally equivalent to $\tilde{\omega}$. Then $\omega = \varphi\tilde{\omega}$, hence $d\omega = d\varphi \wedge \tilde{\omega} = \frac{d\varphi}{\varphi} \wedge \omega$.

DEFINITION: A Hermitian manifold (M, I, ω) is called **locally conformally Kähler (LCK)** if $d\omega = \theta \wedge \omega$, for some closed 1-form θ.

REMARK: This definition is equivalent to the one given on the previous slide.

DEFINITION: The form θ is called **the Lee form** of an LCK-manifold.

CLAIM: If $\dim_{\mathbb{C}} M > 2$, and $d\omega = \theta \wedge \omega$, **then θ is always closed.**

Proof: $0 = d^2\omega = d\theta \wedge \omega$, but $\cdot \wedge \omega : \Lambda^2(M) \longrightarrow \Lambda^4(M)$ **is always injective for $\dim_{\mathbb{C}} M > 2$.** ■

Globally conformally Kähler LCK manifolds

DEFINITION: An LCK manifold (M, I, ω, θ) is called **globally conformally Kähler** if its Lee form θ is exact.

REMARK: If $\theta = d\varphi$, then $e^{-\varphi}\omega$ is Kähler.

PROOF: Indeed, $d(e^{-\varphi}\omega) = e^{-\varphi}\omega \wedge \theta - e^{-\varphi}\omega \wedge d\varphi = 0$. ■

REMARK: By the same argument, **if $\theta = \theta' + d\varphi$, then $e^{-\varphi}\omega$ has the Lee form θ'.** This gives:

CLAIM: Let (M, I, ω, θ) be an LCK manifold. **Then $(M, I, e^{\psi}\omega)$ is also LCK, and its Lee form lies in the same cohomology class.** Conversely, **for each 1-form θ' in the cohomology class of θ, there exists an LCK manifold $(M, I, e^{\psi}\omega)$ with θ' its Lee form.**

LCK structures on Kähler manifolds

THEOREM: (Izu Vaisman) **Any LCK structure (M, I, ω, θ) on a compact Kähler manifold is globally conformally Kähler.**

Proof. Step 1: By a conformal change of ω, we may assume that θ is a sum of a holomorphic and an antiholomorphic form. **Then $d(I\theta) = 0$.**

Step 2: Unless $\theta = 0$ **(and then ω is Kähler),**

$$d(\omega^{n-1}) \wedge I(\theta) = (n-1)\omega \wedge \theta \wedge I(\theta)$$

is a non-zero positive form. **Therefore, $\int_M d(\omega^{n-1}) \wedge I(\theta) > 0$.**

Step 3: Since $I(\theta)$ is closed, $d(\omega^{n-1}) \wedge I(\theta) = d(\omega^{n-1} \wedge I(\theta))$ is exact, hence **Stoke's implies $\int_M d(\omega^{n-1}) \wedge I(\theta) = 0$.** Step 2 implies then that $\theta = 0$. ■

REMARK: We obtain the following corollary. Let (M, ω, θ) be a compact LCK manifold. **Then either θ is exact, and then M is Kähler, or θ is non-exact, and then M cannot admit any Kähler structure.**

REMARK: Further on, **we consider only LCK manifolds where θ is non-exact.**

Vaisman manifolds

REMARK: Vaisman conjectured that all non-Kähler LCK manifolds have b_1 odd. **A counterexample was found by Oeljeklaus and Toma in 2005.**

REMARK: Another conjecture of Vaisman: **No non-Kähler compact LCK manifold can be homotopy equivalent to a compact Kähler manifold.** Still unknown.

For a subclass of LCK manifold, the topology is well understood.

DEFINITION: An LCK manifold (M, ω, θ) is called **Vaisman** ("generalized Hopf") if $\nabla_{LC}\theta = 0$, where ∇_{LC} is the Levi-Civita connection.

THEOREM: (Kamishima-Ornea, 2001) If M is compact, **Vaisman is equivalent to M admitting a conformal holomorphic flow,** acting non-isometrically on its Kähler covering.

EXAMPLE: A Hopf manifold $\mathbb{C}^n \backslash 0/\langle A\rangle$ is Vaisman. It is isometric to $S^{2n-1} \times S^1$, and the Lee field is $\frac{d}{dt}$.

Structure theorem for Vaisman manifolds

DEFINITION: Let X be a projective orbifold, and L an ample line bundle on X. Assume that the total space of L is smooth outside of the zero divizor. The **algebraic cone $\mathcal{C}(X,L)$ of X,L** is the space of all non-zero vectors in L^*.

THEOREM: (Ornea-V.) **Any Vaisman manifold is diffeomorphic to a quotient of $\mathcal{C}(X,L)$ by $x \sim qx$,** where $q \in \mathbb{C}$, $|q| > 1$.

PROPOSITION: A quotient $\mathcal{C}(X,L)/(x \sim qx)$ is always Vaisman.

Proof. Step 1: Let h denote a metric of negative curvature on L^*, and $\varphi(v) := h(v,v)$ be the corresponding function on $\mathcal{C}(X,L)$. Denote by d^c the differential $d^c := IdI^{-1}$. **Then $dd^c\varphi$ is a Kähler form on $\mathcal{C}(X,L)$** (a local calculation).

Step 2: The map $v \longrightarrow qv$ is conformal with respect to this metric. **Therefore, $M := \mathcal{C}(X,L)/(x \sim qx)$ is LCK.**

Step 3: The natural $\mathbb{C}^*$-action is conformal, hence **by Kamishima-Ornea M is Vaisman.** ■

Kähler potentials

DEFINITION: Let (M,I,ω) be a Kähler manifold. A **Kähler potential** is a function satisfying $dd^c\psi=\omega$. Locally, a Kähler potential always exists, and it is unique up to adding real parts of holomorphic functions.

OBSERVATION: Let (M,ω,θ) be a Vaisman manifold, and $(\tilde M,\tilde\omega)\xrightarrow{\pi}M$ be its Kähler covering. Then $\pi^*\theta$ is exact on $\tilde M$: $\pi^*\theta=d\nu$. Moreover, **the function $\psi:=e^{-\nu}$ is a Kähler potential: $dd^c\psi=\tilde\omega$.**

OBSERVATION: Let $\gamma\in\Gamma$ be any element. Since Γ preserves θ, we have $\gamma^*\nu=\nu+c_\gamma$, where c_γ is a constant. Then $\gamma^*\psi=e^{-c_\gamma}\psi$ **(automorphic property).**

COROLLARY: A small deformation of a Vaisman manifold is LCK.

PROOF: Let $\tilde M$ be a Kähler covering of (M,I), $M=\tilde M/\Gamma$, and ψ an automorphic potential on $\tilde M$. Consider a small deformation I' of I. **Then $dI'dI'^{-1}\psi$ is also a positive (1,1)-form.** It is automorphic, hence **Γ acts on $(\tilde M,I,dI'dI'^{-1}\psi)$ by holomorphic homotheties.** ■.

LCK manifold: deformational stability

THEOREM: (Kodaira) **A small deformation of a Kähler manifold is again Kähler.**

REMARK: Not true for Vaisman or LCK manifolds.

THEOREM: (Belgun, Tricerri) Inoue surfaces of type $S^+_{N,p,q,r,t}$ **admit an LCK structure when $t \in \mathbb{R}$ and do not admit it when $t \in \mathbb{C} \backslash \mathbb{R}$.**

DEFINITION: Let $A \in \text{End}(\mathbb{C}^n)$ be a matrix with all eigenvalues $|\alpha_i| < 1$, and $H := \mathbb{C}^n \backslash 0 / \langle A \rangle$ the quotient by the group generated by A. Then H is called **a linear Hopf manifold**.

THEOREM: **A linear Hopf manifold is Vaisman if and only if A is diagonalizable.**

REMARK: Therefore, **neither LCK manifolds nor Vaisman manifolds are stable under small deformations.**

LCK manifolds with potential

DEFINITION: Let (M,ω,θ) be a compact LCK manifold, $(\tilde{M},\tilde{\omega})$ its Kähler covering, Γ the deck transform group, $M = \tilde{M}/\Gamma$. and $\psi \in C^\infty \tilde{M}$ a Kähler potential, $\psi > 0$. Assume that for any $\gamma \in \Gamma$, $\gamma^*\psi = c_\gamma \psi$, for some constant c_γ. Then ψ is called **an automorphic potential** of M, and M **an LCK manifold with potential**.

EXAMPLE: All Hopf manifolds (including non-Vaisman Hopf) are LCK with potential. **All LCK manifolds with potential are small deformations of Vaisman** (Ornea-V., 2009). Therefore, **LCK with potential are diffeomorphic to quotients of algebraic cones by the standard $\mathbb{Z}$-action.**

PROPOSITION: (Ornea-V., 2010) Let (M,ω,θ) be an LCK manifold with an automorphic potential. **Then it has a Kähler covering with monodromy $\mathbb{Z}$.**

THEOREM: A small deformation of an LCK manifold with potential **is again an LCK manifold with potential.**

PROOF: Let $\tilde{M}$ be a Kähler covering of (M,I), $M = \tilde{M}/\Gamma$, and ψ an automorphic potential on $\tilde{M}$. Consider a small deformation I' of I. **Then $dI'dI'^{-1}\psi$ is also a positive (1,1)-form.** It is automorphic, hence Γ **acts on $(\tilde{M},I,dI'dI'^{-1}\psi)$ by holomorphic homotheties.** ■.

Stein manifolds

DEFINITION: A complex variety M is called **holomorphically convex** if for any infinite discrete subset $S \subset M$, there exists a holomorphic function $f \in \mathcal{O}_M$ which is unbounded on S.

DEFINITION: A complex variety is called **Stein** if it is holomorphically convex, and has no compact complex subvarieties.

REMARK: Equivalently, **a complex variety is Stein if it admits a closed holomorphic embedding into $\mathbb{C}^n$.**

THEOREM: (K. Oka, 1942) **A complex manifold M is Stein** if and only M admits a Kähler metric with a **Kähler potential which is positive and proper** (proper = preimages of compact sets are compact).

THEOREM: (Rossi 1965, Andreotti-Siu 1970) Let M be a complex manifold with a boundary, $\dim_{\mathbb{C}} M > 2$, and φ a proper Kähler potential on M, taking values in $[c, \infty[$, and equal to c in the boundary of M. **Then there exists a Stein variety M_c** with isolated singularities, containing M, and it is unique.

Embedding an LCK manifold into a linear Hopf

THEOREM: Let (M, ω, θ) be an LCK manifold with potential, $\dim_{\mathbb{C}} M > 2$, and $\tilde{M}$ is its Kähler covering. Then $\tilde{M}$ **can be compactified by adding a single point to its origin, and the resulting variety is Stein.** Moreover, the monodromy Γ acts on $\tilde{M}$ by holomorphic automorphisms.

PROOF: Follows from Rossi-Andreotti-Siu theorem (we glue in the hole left by excising the set of points where $\psi \leqslant c$). ■

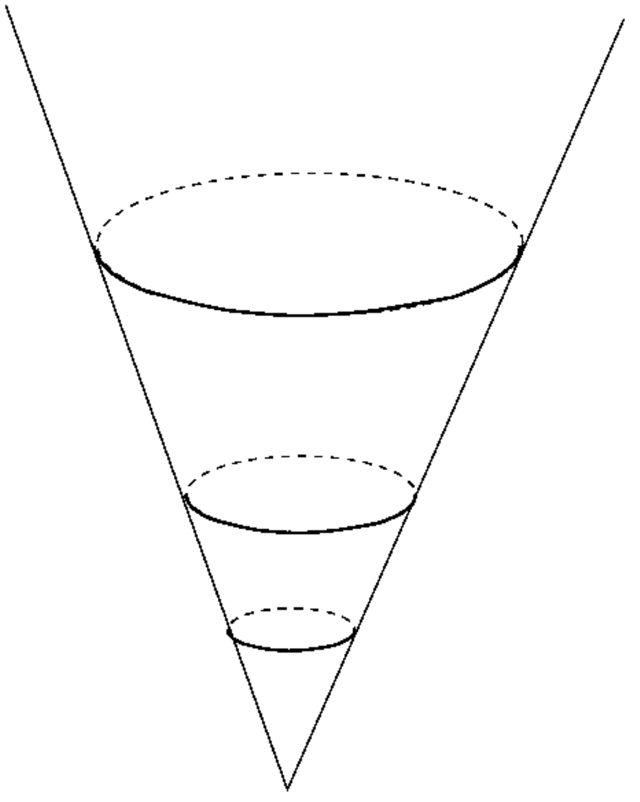
COROLLARY: An LCK manifold with potential admits a holomorphic embedding into a Hopf manifold.

PROOF: A holomorphic embedding into a Hopf manifold is the same as an automorphic embedding into $\mathbb{C}^n$. Using the Stein property, we find a suitable space $V \subset \mathcal{O}_{\tilde{M}}$ preserved by Γ. This gives a map $\tilde{M}/\Gamma \longrightarrow (V \backslash 0)/\Gamma$. ■

This is an LCK analogue of Kodaira's embedding result!

REMARK: Converse is also true: any complex subvariety of a Hopf manifold admits an LCK potential. Indeed, **a subvariety of an LCK manifold with potential is again an LCK manifold with potential.**

A picture of an algebraic cone



A picture of an algebraic cone, with the fundamental domains of the $\mathbb{Z}$-action marked. **Each of these domains has two components of the boundary: strictly pseudoconvex and strictly pseudoconcave.** The pseudoconcave component is filled in, using Rossi-Andreotti-Siu.

Morse-Novikov class of an LCK manifold

DEFINITION: Let (M,ω,θ) be an LCK manifold, and

$$d_\theta := d - \theta : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$$

the "Morse-Novikov" differential on differential forms. Its cohomology $H^i_\theta(M)$ are called **the Morse-Novikov cohomology** of M.

DEFINITION: Let (M,ω,θ) be an LCK manifold, and L a trivial line bundle, with flat connection defined as $\nabla := \nabla_0 + \theta$, where ∇_0 is the trivial connection. Then L is called **the weight bundle** of M.

REMARK: The cohomology of the local system (L,∇) is naturally identified with $H^i_\theta(M)$.

DEFINITION: Clearly, $d_\theta\omega = 0$. Its cohomology class $[\omega] \in H^2_\theta(M)$ is called **the Morse-Novikov class of M**.

REMARK: The Morse-Novikov class is an LCK analogue of a Kähler class.

Automorphic forms of an LCK manifold

Let $(\tilde M, \tilde\omega)$ be a Kähler covering of an LCK manifold $M = \tilde M/\Gamma$. Consider the character of Γ, defined through the scale factor of $\tilde\omega$: $\gamma^*\tilde\omega = \chi(\gamma)\tilde\omega, \quad \forall \gamma\in\Gamma$.

DEFINITION: A differential form α on $\tilde M$ is called **automorphic** if $\gamma^*\alpha = \chi(\gamma)\alpha$, where $\chi: \Gamma \longrightarrow \mathbb{R}^{>0}$ is the character of Γ defined above.

REMARK: An automorphic form on $\tilde M$ is the same as L-valued form on M.

DEFINITION: Let M be an LCK manifold, $\Lambda^{1,1}_{\chi,d}(\tilde M)$ the space of closed, automorphic $(1,1)$-forms on its Kähler covering $\tilde M$, and $\mathcal{C}^\infty_\chi(\tilde M)$ be the space of automorphic functions on $\tilde M$. Consider the quotient

$$H^{1,1}_{BC}(M,L) := \frac{\Lambda^{1,1}_{\chi,d}(\tilde M)}{dd^c(\mathcal{C}^\infty_\chi(\tilde M))}.$$

This group is finite-dimensional. It is called **the Bott-Chern cohomology group of an LCK manifold**.

Bott-Chern class of an LCK manifold

DEFINITION: The Kähler form $\tilde\omega$ on $\tilde M$ is obviously closed and automorphic. Its cohomology class $[\tilde\omega] \in H^{1,1}_{BC}(M, L)$ is called **the Bott-Chern class of M**.

REMARK: It is a holomorphic version of a Morse-Novikov class.

A tautological claim: An LCK manifold admits automorphic potential if and only if its Bott-Chern class vanishes.

A cohomological version of the LCK embedding theorem: Let M be an LCK manifold. Then M **admits a complex embedding to a Hopf manifold iff its Bott-Chern class vanishes.**

Morse-Novikov and Bott-Chern class

REMARK: The Bott-Chern cohomology group is $H^{1,1}_{BC}(M,L) := \Lambda^{1,1}_{\chi,d}(\tilde M)/dd^c(\mathcal{C}^\infty_\chi(\tilde M))$, and the Morse-Novikov is $H^2_\theta(M) := \Lambda^2_{\chi,d}/d(\Lambda^1_{\chi,d})$. This gives a natural map $H^{1,1}_{BC}(M,L) \longrightarrow H^2_\theta(M)$ **mapping the Bott-Chern class to Morse-Novikov class**.

PROBLEM: Morse-Novikov class is very easy to compute, because it's topological invariant. Bott-Chern class is hard to compute. Can we express Bott-Chern through Morse-Novikov?

CONJECTURE: (an LCK dd^c-lemma) Let M be an LCK-manifold, and η an automorphic (1,1)-form on M, with $\eta = d\rho$ for an automorphic 1-form ρ. **Then $\eta = dd^c\nu$, where ν is an automorphic function.**

REMARK: If this is true, we would be able to find **many new examples of complex manifolds which are not LCK.** Indeed, the Morse-Novikov cohomology of (M,L) often vanishes, but then the dd^c-lemma would imply that M is LCK with potential, hence diffeomorphic to Vaisman.

REMARK: Suppose that the LCK dd^c-lemma is false. Then $\tilde M$ admits a holomorphic line bundle B, representing a non-torsion, infinitely divisible element in $Pic(\tilde M)$, such that the monodromy action γ satisfies $\gamma^*(B) \cong B \otimes B$. **Bizzarre!**