Bogomolov-Tian-Todorov for holomorphic symplectic manifolds

Misha Verbitsky

Cable car Algebra seminar,

University of Haifa,

June 20, 2024

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$. **The eigenvalues of this operator are $\pm\sqrt{-1}$.** The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the standard one.

CLAIM: (the Hodge decomposition determines the complex structure)
Let M be a smooth $2n$-dimensional manifold. **Then there is a bijective correspondence between the set of almost complex structures, and the set of sub-bundles $T^{0,1}M \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ satisfying $\dim_{\mathbb{C}} T^{0,1}M = n$ and $T^{0,1}M \cap TM = 0$** (the last condition means that there are no real vectors in $T^{1,0}M$, that is, that $T^{0,1}M \cap T^{1,0}M = 0$).

Proof: Set $I\big|_{T^{1,0}M} = \sqrt{-1}$ and $I\big|_{T^{0,1}M} = -\sqrt{-1}$. ■

Hodge theory

DEFINITION: Let (M, I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M, I)$ of (p, q)-forms on (M, I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\bar{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

DEFINITION: A manifold is called **Kähler** if it equipped with a closed real (1,1)-form ω such that $\omega(Ix, x) > 0$ for any non-zero vector x.

THEOREM: ("Hodge decomposition on cohomology") Let M be a compact Kähler manifold. **Then any cohomology class can be represented as a sum of closed (p, q)-forms.**

Schouten brackets

DEFINITION: Let M be a complex manifold, and $\Lambda^{0,p}(M) \otimes T^{1,0}M$ the sheaf of $T^{1,0}M$-valued $(0,p)$-forms. Consider the commutator bracket $[\cdot,\cdot]$ on $T^{1,0}M$, and let $\overline{\mathcal{O}}_M$ denote the sheaf of antiholomorphic functions. Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, it is naturally extended to $\Lambda^{0,p}(M) \otimes_{C^\infty M} T^{1,0}M = \overline{\Omega^p M} \otimes_{\overline{\mathcal{O}}_M} T^{1,0}M$, giving a bracket

$$[\cdot,\cdot]: \ \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

This bracket is called **Schouten bracket**.

REMARK: Since $[\cdot,\cdot]$ is $\overline{\mathcal{O}}_M$-linear, the Schouten bracket satisfies the Leibnitz identity:

$$\overline{\partial}([\alpha,\beta]) = [\overline{\partial}\alpha,\beta] + [\alpha,\overline{\partial}\beta].$$

This allows one to extend the Schouten bracket to the $\overline{\partial}$-cohomology of the complex $(\Lambda^{0,*}(M) \otimes T^{1,0}M, \overline{\partial})$, which coincide with the cohomology of the sheaf of holomorphic vector fields: $[\cdot,\cdot]: \ H^p(TM) \times H^q(TM) \longrightarrow H^{p+q}(TM)$.

Maurer-Cartan equation and deformations

CLAIM: Let (M,I) be an almost complex manifold, and B an abstract vector bundle over $\mathbb{C}$ isomorphic to $\Lambda^{0,1}(M)$. Consider a differential operator $\overline{\partial}: C^\infty M \longrightarrow B = \Lambda^{0,1}(M)$ satisfying the Leibnitz rule. Its symbol is a linear map $u: \Lambda^1(M,\mathbb{C}) \longrightarrow B$. Then $B = \frac{\Lambda^1(M,\mathbb{C})}{\ker u} = \Lambda^{0,1}(M)$. Extend $\overline{\partial}: C^\infty M \longrightarrow B$ to the corresponding exterior algebra using the Leibnitz rule:

$$C^\infty M \xrightarrow{\overline{\partial}} B \xrightarrow{\overline{\partial}} \Lambda^2 B \xrightarrow{\overline{\partial}} \Lambda^3 B \xrightarrow{\overline{\partial}} ...$$

Then integrability of I is equivalent to $\overline{\partial}^2 = 0$.
Proof: This is essentially the Newlander-Nirenberg theorem. ■

REMARK: Almost complex deformations of I are given by the sections $\gamma \in T^{1,0}M \otimes \Lambda^{0,1}(M)$, with the integrability relation $(\overline{\partial}+\gamma)^2 = 0$ rewritten as **the Maurer-Cartan equation** $\overline{\partial}(\gamma) = -\{\gamma,\gamma\}$. Here $\overline{\partial}(\gamma)$ is identified with the anticommutator $\{\overline{\partial},\gamma\}$, and $\{\gamma,\gamma\}$ is anticommutator of γ with itself, where γ is considered as a $\Lambda^{0,1}(M)$-valued differential operator. **This identifies $\{\gamma,\gamma\}$ with the Schouten bracket.**

REMARK: We shall write $[\gamma,\gamma]$ instead of $\{\gamma,\gamma\}$, because this usage is more common.

Solving the Maurer-Cartan equation recursively

DEFINITION: The Kuranishi deformation space, can be defined as **the space of solutions of Maurer-Cartan equation $\overline{\partial}(\gamma) = -[\gamma, \gamma]$ modulo the diffeomorphism action.**

DEFINITION: Write γ as power series, $\gamma = \sum_{i=0}^{\infty} t^{i+1}\gamma_i$. **Then the Maurer-Cartan becomes**

$$\overline{\partial}\gamma_0 = 0, \quad \overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

We say that deformations of complex structures are **unobstructed** if the solutions $\gamma_1, ..., \gamma_n, ...$ of (**) can be found for γ_0 in any given cohomology class $[\gamma_0] \in H^1(M, TM)$.

REMARK 1: Notice that **the sum $\sum_{i+j=p-1}[\gamma_i, \gamma_j]$ is always $\overline{\partial}$-closed.** Indeed, the Schouten bracket commutes with $\overline{\partial}$, hence

$$\overline{\partial} \sum_{i+j=p-1} [\gamma_i, \gamma_j] = -\sum_{i+j+k=p-1} [\gamma_i, [\gamma_j, \gamma_k]] + [[\gamma_i, \gamma_j], \gamma_k]. \quad (***)$$

vanishes as a sum of triple supercommutators. **Obstructions to deformations** are given by cohomology classes of the sums $\sum_{i+j=p-1}[\gamma_i, \gamma_j]$, which are defined inductively. These classes are called **Massey powers** of γ_0.

Tian-Todorov lemma

DEFINITION: Assume that M is a complex n-manifold with trivial canonical bundle K_M, and Φ a non-degenerate section of K_M. We call a pair (M, Φ) **a Calabi-Yau manifold**. Substitution of a vector field into Φ gives an isomorphism $TM \cong \Omega^{n-1}(M)$. Similarly, one obtains an isomorphism

$$\Lambda^{0,q}M \otimes \Lambda^p TM \longrightarrow \Lambda^{0,q}M \otimes \Lambda^{n-p,0}M = \Lambda^{n-q,p}M. \qquad (*)$$

Yukawa product $\bullet: \Lambda^{p,q}M \otimes \Lambda^{p_1,q_1}M \longrightarrow \Lambda^{p+p_1-n,q+q_1}M$ is obtained from the usual product

$$\Lambda^{0,q}M \otimes \Lambda^p TM \times \Lambda^{0,q_1}M \otimes \Lambda^{p_1}TM \longrightarrow \Lambda^{0,q+q_1}M \otimes \Lambda^{p+p_1}TM$$

using the isomorphism (*).

TIAN-TODOROV LEMMA: Let (M, Φ) be a Calabi-Yau manifold, and

$$[\cdot,\cdot]: \Lambda^{0,p}(M) \otimes T^{1,0}M \times \Lambda^{0,q}(M) \otimes T^{1,0}M \longrightarrow \Lambda^{0,p+q}(M) \otimes T^{1,0}M.$$

its Schouten bracket. Using the isomorphism (*), we can interpret Schouten bracket as a map

$$[\cdot,\cdot]: \Lambda^{n-1,p}(M) \times \Lambda^{n-1,q}(M) \longrightarrow \Lambda^{n-1,p+q}(M).$$

Then, for any $\alpha \in \Lambda^{n-1,p}(M)$, $\beta \in \Lambda^{n-1,p_1}(M)$, one has

$$[\alpha, \beta] = \partial(\alpha \bullet \beta) - (\partial\alpha) \bullet \beta - (-1)^{n-1+p} \alpha \bullet (\partial\beta),$$

where $\bullet$ denotes the Yukawa product.

dd^c-lemma

DEFINITION: Let M be a complex manifold, and $I: TM \longrightarrow TM$ its complex structure operator. **The twisted differential** of M is $IdI^{-1}: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$, where I acts on 1-forms as an operator dual to $I: TM \longrightarrow TM$, and on the rest of differential forms multiplicatively.

REMARK: Consider the Hodge decomposition of the de Rham differential, $d = \partial + \overline{\partial}$, where $\partial: \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$ and $\overline{\partial}: \Lambda^{p,q}(M,I) \longrightarrow \Lambda^{p+1,q}(M,I)$. **Then $d = \operatorname{Re}\partial$ and $d^c = \operatorname{Im}\partial$.** Also, $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

THEOREM: (**dd^c-lemma**) Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
Then η is dd^c-exact, that is, $\eta \in \operatorname{im} dd^c$. Equivalently, **if η is ∂-exact and $\overline{\partial}$-closed, it is dd^c-exact.**

REMARK: This statement is weaker that the Kähler condition, but it immediately implies almost every cohomological property of Kähler manifolds, except the Lefschetz $\mathfrak{sl}(2)$-action. In particular, **dd^c-lemma is sufficient to prove the Bogomolov-Tian-Todorov theorem**, claiming that the deformations of Calabi-Yau manifolds are unobstructed.

Bogomolov-Tian-Todorov theorem

THEOREM: Let M be a compact complex n-manifold with trivial canonical bundle which satisfies dd^c-lemma. **Then its deformations are unobstructed**.

Proof. Step 1: Let's start with a cohomology class $[\gamma_0] \in H^1(TM) = H^1(\Omega^{n-1}M)$. To prove that the deformations are unobstructed, we need to solve the equation system

$$\overline{\partial}\gamma_0 = 0, \quad \overline{\partial}\gamma_p = -\sum_{i+j=p-1} [\gamma_i, \gamma_j]. \quad (**)$$

recursively, starting from a representative γ_0 of $[\gamma_0]$. Identifying $\Lambda^{0,1}(T^{1,0}M)$ with $\Lambda^{0,1}(\Lambda^{n-1,0}M) = \Lambda^{n-1,1}(M)$, **we choose a representative $\gamma_0 \in \Lambda^{n-1,1}(M)$ of $[\gamma_0]$ which is ∂ and $\overline{\partial}$-closed;** this is possible to do using $\partial\overline{\partial}$-lemma (in Kähler situation, take a harmonic representative).

Step 2: Using induction, we may assume that (**) is solved up to γ_{n-1}, and, moreover, the solutions satisfy $\partial\gamma_i = 0$. By Tian-Todorov lemma,

$$\alpha := [\gamma_i, \gamma_j] = \partial(\gamma_i \bullet \gamma_j) - (\partial\gamma_i) \bullet \gamma_j - (-1)^{n-1+p}\gamma_i \bullet (\partial\gamma_j) = \partial(\gamma_i \bullet \gamma_j),$$

hence it is ∂-exact; as shown in Remark 1 above, it is also $\overline{\partial}$-closed. By dd^c-lemma, α is $\partial\overline{\partial}$-exact. This implies that $-\sum_{i+j=n-1}[\gamma_i, \gamma_j] = \overline{\partial}\partial\beta$. **Taking $\gamma_n := \partial\beta$, we obtain a solution of (**) which is also ∂-closed, hence satisfy the induction assumptions.** ∎

Holomorphically symplectic manifolds

DEFINITION: Let (M,I) be a complex manifold, and $\Omega\in\Lambda^2(M,\mathbb{C})$ a differential form. We say that Ω is **non-degenerate** if $\ker\Omega\cap T_{\mathbb{R}}M=0$. We say that it is **holomorphically symplectic** if it is non-degenerate, $d\Omega=0$, and $\Omega(IX,Y)=\sqrt{-1}\,\Omega(X,Y)$.

REMARK: The equation $\Omega(IX,Y)=\sqrt{-1}\Omega(X,Y)$ means that **Ω is complex linear with respect to the complex structure on $T_{\mathbb{R}}M$ induced by I.**

REMARK: Consider the Hodge decomposition $T_{\mathbb{C}}M=T^{1,0}M\oplus T^{0,1}M$ (decomposition according to eigenvalues of I). Since $\Omega(IX,Y)=\sqrt{-1}\,\Omega(X,Y)$ and $I(Z)=-\sqrt{-1}\,Z$ for any $Z\in T^{0,1}(M)$, we have $\ker(\Omega)\supset T^{0,1}(M)$. Since $\ker\Omega\cap T_{\mathbb{R}}M=0$, real dimension of its kernel is at most $\dim_{\mathbb{R}}M$, giving $\dim_{\mathbb{R}}\ker\Omega=\dim M$. **Therefore,** $\ker(\Omega)=T^{0,1}M$**.**

COROLLARY: Let Ω be a holomorphically symplectic form on a complex manifold (M,I). **Then I is determined by Ω uniquely.**

C-symplectic structures

DEFINITION: (Bogomolov, Deev, V.) Let M be a smooth $4n$-dimensional manifold. A complex-valued form Ω on M is called **almost C-symplectic** if $\Omega^{n+1} = 0$ and $\Omega^n \wedge \overline{\Omega}^n$ is a non-degenerate volume form. It is called **C-symplectic** when it is also closed.

THEOREM: Let $\Omega \in \Lambda^2(M, \mathbb{C})$ be a C-symplectic form, and $T^{0,1}_\Omega(M)$ be equal to $\ker \Omega$, where $\ker \Omega := \{v \in TM \otimes \mathbb{C} \ \mid \ \Omega \lrcorner v = 0\}$. Then $T^{0,1}_\Omega(M) \oplus \overline{T^{0,1}_\Omega(M)} = TM \otimes_{\mathbb{R}} \mathbb{C}$, hence **the sub-bundle $T^{0,1}_\Omega(M)$ defines an almost complex structure I_Ω on M.** If, in addition, **Ω is closed, I_Ω is integrable, and Ω is holomorphically symplectic on (M, I_Ω).**

Proof: Rank of Ω is $2n$ because $\Omega^{n+1} = 0$ and $\operatorname{Re}\Omega$ is non-degenerate. Then $\ker \Omega \oplus \overline{\ker \Omega} = T_{\mathbb{C}}M$. The relation $[T^{0,1}_\Omega(M), T^{0,1}_\Omega(M)] \subset T^{0,1}_\Omega(M)$ follows from Cartan's formula

$$d\Omega(X_1, X_2, X_3) = \frac{1}{6} \sum_{\sigma \in \Sigma_3} (-1)^{\tilde{\sigma}} \operatorname{Lie}_{X_{\sigma_1}} \Omega(X_{\sigma_2}, X_{\sigma_3}) + (-1)^{\tilde{\sigma}} \Omega([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3})$$

which gives, for all $X, Y \in T^{0,1}M$, and any $Z \in TM$,

$$d\Omega(X, Y, Z) = \Omega([X, Y], Z),$$

implying that $[X, Y] \in T^{0,1}M$. ■

Local Torelli theorem

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient CTeich $:= \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map CTeich $\longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

DEFINITION: Let M be a compact complex manifold. We say that M **satisfies $\partial\bar\partial$-lemma in term $\Lambda^{p,q}(M)$** if any ∂-closed, $\bar\partial$-exact (p,q)-form belongs to the image of $\partial\bar\partial$.

THEOREM: ("Local Torelli theorem")
Let (M, Ω) be a C-symplectic manifold. Assume that $H^{0,1}(M) = 0$, $H^{2,0}(M) = \mathbb{C}$. Assume also that M satisfies $\partial\bar\partial$-lemma in $\Lambda^{1,2}(M)$ and has Hodge decomposition in $H^2(M)$. Let $W := \frac{H^2(M,\mathbb{C})}{\langle\overline{\Omega}\rangle}$. Then the period map composed with the natural projection $H^2(M, \mathbb{C}) \mapsto W$ **defines a local difeomorphism from CTeich to a neighbourhood of 0 in W.**

Proof: Surjectivity is Kurnosov-V., injectivity: Soldatenkov-V. ("holomorphic Moser lemma").

Holomorphically symplectic Moser's lemma

DEFINITION: Let (M, I, Ω) be a holomorphically symplectic manifold, and CSymp the space of all C-symplectic forms. The quotient $\text{CTeich} := \frac{\text{CSymp}}{\text{Diff}_0}$ is called **the holomorphically symplectic Teichmüller space**, and the map $\text{CTeich} \longrightarrow H^2(M, \mathbb{C})$ taking (M, I, Ω) to the cohomology class $[\Omega] \in H^2(M, \mathbb{C})$ **the holomorphically symplectic period map**.

The period map is locally an embedding. This is immediately implied by the following version of Moser's lemma.

THEOREM: (Soldatenkov, V.)
Let (M, I_t, Ω_t), $t \in [0, 1]$ be a family of C-symplectic forms on a compact manifold. Assume that the cohomology class $[\Omega_t] \in H^2(M, \mathbb{C})$ is constant, and $H^{0,1}(M, I_t) = 0$, where $H^{0,1}(M, I_t) = H^1(M, \mathcal{O}_{(M,I_t)})$ is cohomology of the sheaf of holomorphic functions. Then **there exists a smooth family of diffeomorphisms $V_t \in \text{Diff}_0(M)$, such that $V_t^* \Omega_0 = \Omega_t$.**

Local Torelli theorem for a K3 surface

REMARK: In real dimension 4, C-symplectic form **is a pair ω_1, ω_2 of symplectic forms which satisfy $\omega_1^2 = \omega_2^2$ and $\omega_1 \wedge \omega_2 = 0$.**

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof: Next slide

REMARK: This theorem locally identifies $H^{1,1}(M)$ with the neighbourhood Ω in the C-symplectic Teichmüller space, proving that it is smooth and $b_2 - 2$-dimensional. **This proves the local Torelli theorem for K3.**

REMARK: The proof of this theorem is done using the same argument as used to prove the Maurer-Cartan equation, central to Kuranishi theory. Indeed, **the equation (*) we are going to solve below is a version of Maurer-Cartan, adopted and simplified for the C-symplectic structures.**

Local Torelli theorem for K3 (2)

THEOREM: Let (M, I, Ω) be a complex holomorphically symplectic surface with $H^{0,1}(M) = 0$, that is, a K3 surface. Then for any sufficiently small cohomology class $[\eta] \in H^{1,1}(M)$, **there exists a C-symplectic form $\Omega + \rho$, where $\rho \in \Lambda^{1,1}M + \Lambda^{0,2}M$ is a closed form which satisfies $\rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, and $\rho^{1,1}$ is ∂-cohomologous to $[\eta]$.** Moreover, the cohomology class of ρ **is uniquely determined by $[\eta]$**.

Proof. Step 1: Since $(\Omega+\rho)^2 = \rho^{1,1} \wedge \rho^{1,1} = -\Omega \wedge \rho^{0,2}$, this form is (almost) C-symplectic. **To prove that it is C-symplectic, we need to find ρ such that that $d\rho = 0$.**

Step 2: From Hodge to de Rham isomorphism, we obtain that the cohomology class $[u]$ of $\Omega + \rho$ is equal to $[\Omega + \eta + u^{0,2}]$. Since M is K3, we have $H^{0,2}(M) = \mathbb{C}[\overline{\Omega}]$, which gives $[u^{0,2}] = \lambda[\overline{\Omega}]$, for some $\lambda \in \mathbb{C}$. since $(\Omega+\rho)^2 = 0$, this gives $[\Omega \wedge u^{0,2}] = [\eta]$. Then $\lambda = -\frac{[\eta^2]}{[\Omega\wedge\overline{\Omega}]}$. **We proved that the cohomology class of $\Omega + \rho$ is uniquely determined by η.**

Local Torelli theorem for K3 (3)

Below, we need the following version of **$\partial\bar\partial$-lemma**: **for any $(1,2)$-form α, which is ∂-exact and $\bar\partial$-closed, $\alpha = \bar\partial\beta$, where β is ∂-exact.**

Step 3: Let Λ_Ω be contraction with the $(2,0)$-bivector associated with Ω. This operation clearly commutes with $\bar\partial$. Then $\rho^{1,1}\wedge\rho^{1,1} = -\Omega\wedge\rho^{0,2}$ is equivalent to $\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\rho^{0,2}$. **To solve the equation $d\rho = 0$, we solve the equivalent equation, which is a version of Maurer-Cartan**

$$\partial\Lambda_\Omega(\rho^{1,1}\wedge\rho^{1,1}) = -\bar\partial\rho^{1,1}, \qquad \partial\rho^{1,1} = 0. \qquad (*)$$

Let γ_0 be the harmonic (1,1)-form representing $[\eta]$. We solve the equation (*) inductively by taking

$$\bar\partial\gamma_n = \partial\Lambda_\Omega\left(\sum_{i+j=n-1}\gamma_i\wedge\gamma_j\right). \quad (**)$$

Such γ_n is found using $\partial\bar\partial$-lemma, because the RHS of (**) is ∂-exact and $\bar\partial$-closed, which is clear because $\bar\partial$ commutes with Λ_Ω. Since $\bar\partial\sum_i\gamma_i = \partial\Lambda_\Omega\left(\sum_{i,j}\gamma_i\wedge\gamma_j\right)$, the sum $\rho^{1,1} := \sum\gamma_i$ is a solution of (*).

Step 4: Since γ_i, $i>0$ are ∂-exact, the ∂-cohomology class of γ is $[\gamma_0] = [\eta]$.
■