Hyperkahler SYZ conjecture and multiplier ideal sheaves

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HOLOMORPHICALLY SYMPLECTIC VARIETIES AND MODULI SPACES

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A plan of the minicourse

Lecture 1: Moduli spaces, the Kähler cone and the mapping class group

Lecture 2: Multiplier ideal sheaves

Lecture 3: Hyperkaehler SYZ conjecture

Hyperkähler manifolds

Definition: A hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

Definition: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering, which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Calabi-Yau theorem gives a unique Ricci-flat Kähler metric on M, in any Kähler class, if $c_1(M) = 1$. If M is also holomorphically symplectic, this metric is **hyperkähler** (Kähler with respect to an$\mathbb{C}P^1$ of complex structures). It follows from Bochner's vanishing, Berger's classification of irreducible holonomy groups, and de Rham's decomposition theorem.

The Teichmuller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$ the "framed, coarse moduli space" of complex structures. We call it **the Teichmuller space.**

Remark: Teich is **finite-dimensional** (Kodaira), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Lambda := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The coarse moduli space of complex structures on M is a connected component of $\operatorname{Teich}/\Lambda$.

Remark: This terminology is **standard for curves.**

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = q(\eta, \eta)^n$, for some integer quadratic form q on $H^2(M)$.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by this relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$C_{2n-2}^{n-1} q(\eta, \eta) = (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - (1-n) \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)$$

where Ω is the holomorphic symplectic form.

Remark: q **has signature** $(b_2 - 3, 3)$**.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$ where ω is a Kähler form.

The period map

Definition: Let (M, I) be a simple hyperkaehler manifold, and Teich **a connected component of the set of Kähler points** on its Teichmuller space.

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

Remark: P maps Teich into an open subset of a quadric, defined by

$$W := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

THEOREM: Let M be a simple hyperkaehler manifold, and Teich its Teichmuller space. Then
(i) (Bogomolov) **The period map P: Teich $\longrightarrow W$ is etale.**
(ii) (Huybrechts) It is **surjective.**

Remark: Bogomolov's theorem implies that Teich **is smooth.**

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact simply connected (or nilpotent) Kaehler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma$ has finite kernel, and its image has finite index in Γ.**

Theorem: Let M be a simple hyperkaehler manifold, and Γ as above. Then
(i) $\Gamma\Big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup** of $O(H^2(M,\mathbb{Q}),q)$.
(ii) The map $\Gamma \longrightarrow O(H^2(M,\mathbb{Q}),q)$ **has finite kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ **preserves the Bogomolov-Beauville-Fujiki up to a sign.**

Step 2: The sign is fixed, because Γ preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Computation of the mapping class group (cont.)

Step 3: $O(H^2(M,\mathbb{Q}),q)$ acts on $H^*(M,\mathbb{Q})$ by automorphisms preserving Pontryagin classes (Looijenga-Lunts, V.). Therefore $\Gamma\Big|_{H^2(M,\mathbb{Q})}$ **is arithmetic.**

Step 4: The kernel K of the map $\Gamma \longrightarrow \Gamma\Big|_{H^2(M,\mathbb{Q})}$ is finite, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

Remark: The same argument also proves that **the group of automorphisms of $H^*(M,\mathbb{Q})$ preserving p_1 is projected to $O(H^2(M,\mathbb{Q}),q)$ or $Pin(H^2(M,\mathbb{Q}),q)$ with finite kernel**.

Remark: The center of $Spin(H^2(M,\mathbb{Q}),q)$ **acts on $H^i(M)$ as $(-1)^i$.**

Non-separate points in the Teichmuller space

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separate points, then $P(I_1) = P(I_2)$, and (M, I_1) is birationally equivalent to (M, I_2)

Remark: Whenever hyperkaehler manifolds (M, I_1) and (M, I_2) are birationally equivalent, both of these varieties contain a rational curve (Boucksom). A general hyperkaehler manifold has no curves. Therefore **the period map P is locally bijective for general $I \in$ Teich.**

The Mumford-Tate group

Definition: Let (M,I) be a Kähler manifold, and $\mathcal{I} \in \operatorname{End}(H^*(M,\mathbb{R}))$ **the Hodge decomposition operator** acting on (p,q)-forms as a multiplication by $(p-q)\sqrt{-1}$. Consider the smallest rational Lie subalgebra containing $\mathcal{I}$, and let $G_{MT} \subset GL(H^*(M,\mathbb{R})$ be the corresponding Lie group. It is called **the Mumford-Tate group of (M,I)**,

Remark: The Mumford-Tate group acts on the ring $H^*(M)$ by automorphisms. Indeed, the Lie algebra of derivations of $H^*(M)$ is rational and contains $\mathcal{I}$.

CLAIM: The Mumford-Tate group G_{MT} is a connected component of a group $G \subset \operatorname{Aut}(H^*(M))$ **stabilizing all rational (p,p)-vectors** in the tensor algebra $T^{\otimes}(H^*(M))$.

Proof: Follows from Chevalley's theorem which claims that an algebraic group is determined by its ring of invariants.

The Mumford-Tate generic complex structures

Definition: Let S be a holomorphic family of complex structures of Kaehler type on a compact manifold M. For any rational tensor $v \in T^{\otimes}(H^*(M,\mathbb{Q}))$, denote by $Z_v \subset S$ the set of all $I \in S$ for which v has type (p,p). Let Z be the union of all Z_v of positive codimension. We say that $I \in S$ is **Mumford-Tate generic** if $I \notin Z$.

Remark: (Lower semicontinuity of Mumford-Tate group) $G_{MT}(M,I)$ **is the same for all Mumford-Tate generic** I and, moreover, $G_{MT}(I') \subset G_{MT}(M,I)$ for all $I' \in S$.

Remark: For a simple hyperkaehler manifold, the Lie group of automorphisms of $H^*(M)$ preserving p_1 is $O(H^2(M),q)$, or $Spin(H^2(M),q)$ as shown above. It is known that the set of all $\mathcal{I}$ associated with complex structures generates $\mathfrak{so}(H^2(M),q)$.

Definition: A simple hyperkähler manifold is called **Mumford-Tate generic** if the corresponding Mumford-Tate Lie group is maximal, that is, coincides with $O(H^2(M),q)$ or $Spin(H^2(M),q)$.

Geometry of Mumford-Tate generic complex structures

THEOREM: (V.) Let I be a Mumford-Tate generic complex structure on a simple hyperkaehler manifold, and $Z \subset (M, I)$ a complex subvariety. Then Z **is symplectic outside of its singularities.** Moreover, a normalization $\tilde{Z}$ of Z is holomorphically symplectic.

Corollary: The Mumford-Tate generic hyperkähler manifolds have no curves, and no divisors.

Remark: Actually, Z is trianalytic (complex with respect to a sphere of complex structures).

The Kähler cone and the Mumford-Tate group

THEOREM (Demailly, Paun) Let M be a compact Kaehler manifold, and W the set of real (1,1)-classes $\eta \in H^{1,1}(M,\mathbb{R})$ which satisfy $\int_Z \eta^{\dim Z} > 0$ for any complex analytic subvariety $Z \subset M$. Then one of connected components of W is a Kaehler cone of M.

Corollary: The Kähler cone of a compact Kaehler manifold, **is invariant under the centralizer of $\mathcal{I}$ in the Mumford-Tate group.**

Proof: All integer homology (p,p)-cycles are Mumford-Tate invariant, hence $\int_Z \eta^{\dim Z} = \int_Z \gamma(\eta)^{\dim Z}$ for any $\gamma \in G_{MT}$. ∎

Corollary: The Kaehler cone of a Mumford-Tate generic hyperkaehler manifold **is one of two components of a set $\{\nu \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ q(\nu,\nu) > 0\}$.**

Proof: It is $O(H^{1,1}(M))$ invariant and invariant under multiplication by positive numbers. The group $O(H^2(M,\mathbb{R})) \times \mathbb{R}^{>0}$ acts on $H^{1,1}(M)$ with 5 orbits, which can be listed explicitly.

Nef classes and pseudoeffective classes

Definition: A class $\eta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive current, and **nef** if it lies in a closure of a Kähler cone.

The divisorial Zariski decomposition theorem: (S. Boucksom) Let M be a simple hyperkähler manifold. Then **every pseudoeffective class can be decomposed as a sum**

$$\eta = \nu + \sum_i a_i E_i$$

where ν is nef, a_i positive numbers, and E_i exceptional divisors satisfying $q(E_i, E_i) < 0$. Conversely, **every such sum is pseudoeffective.**

The birational nef cone

Remark: Let M_1, M_2 be holomorphic symplectic manifolds, bimeromorphically equivalent. Then $H^2(M_1)$ **is naturally isomorphic to** $H^2(M_2)$**, and this isomorphism is compatible with Bogomolov-Beauville-Fujiki form.**

Definition: A **modified nef cone** (also "birational nef cone" and "movable nef cone") is a closure of a union of all nef cones for all bimeromorphic models of a holomorphically symplectic manifold M.

THEOREM: (D. Huybrechts, S. Boucksom)
The modified nef cone is dual to the pseudoeffective cone under the Bogomolov-Beauville-Fujiki pairing.

Corollary: Let M be a simple hyperkähler manifold such that all integer $(1,1)$-classes satisfy $q(\nu,\nu) \geqslant 0$. **Then its Kähler cone is one of two components** K_+ **of a set** $K := \{\nu \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ q(\nu,\nu) > 0\}$.

Proof: The pseudoeffective cone of M is contained in K_+ by divisorial Zariski decomposition. Therefore, the modified nef cone K_{MN} contains K_+. This means that $K_{MN} = K_+$. ■