Hyperkähler SYZ conjecture and multiplier ideal sheaves

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HOLOMORPHICALLY SYMPLECTIC VARIETIES AND MODULI SPACES

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Currents and generalized functions

Definition: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x \in M} \left(|f| + |\nabla f| + ... + |\nabla^k f| \right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

Definition: **A generalized function** is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

Definition: **A k-current** is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

Remark: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

Definition: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

Claim: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

Definition: On an complex manifold, **(p, q)-currents** are (p, q)-forms with coefficients in generalized functions

Remark: **In the literature, this is sometimes called $(n-p, n-q)$-currents**.

Claim: The Dolbeault lemma holds on (p, q)-currents, and **the $\overline{\partial}$-cohomology are the same as for forms.**

Positive forms and currents

Definition: A **weakly positive (p,p)-form** onis a real (p,p)-form η which satisfies $\eta(x_1, Ix_1, x_2, Ix_2, ...x_p, Ix_p) \geqslant 0$ for all $x_1, ...n_p \in TM$. **The set of weakly positive (p,p)-forms is a convex cone.**

Definition: **A cone of strongly positive (p,p)-forms** is a convex cone generated by $\eta_1 \wedge \eta_2 \wedge ... \wedge \eta_p$, for all poisitve (1,1)-forms $\eta_1, ..., \eta_p$.

Claim: For $(n-1, n-1)$-forms, strong positivity is the same as weak.

Claim: The cones of strongly and weakly positive forms are dual.

Remark: The 0 form is weakly positive and strongly positive.

Definition: A **strongly/weakly positive (p,p)-current** is a current taking non-negative values on weakly/strongly positive compactly supported $(n-p, n-p)$-forms.

Remark: A positive (p,p)-current is C^0-continuous.

Positive currents and measures

Definition: A **positive generalized function** is a generalized function taking non-negative values on all positive volume forms.

Remark: Positive generalized functions are C^0-continuous. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

Definition: A **mass measure** of a positive (p, p)-current η on a Hermitian n-manifold (M, ω) is a measure $\eta \wedge \omega^{n-p}$. **It is non-negative, and positive, if $\eta \neq 0$.**

Theorem: The space of positive currents with bounded measure is (weakly) compact.

Proof: Follows from precompactness of bounded sets in weak-*-topology.

Remark: Since the space of currents is Montel, **all bounded subsets are precompact.**

Closed positive currents and psh functions

Definition: Let $Z \subset M$ be a complex analytic subvariety. **The current of integration** $[Z]$ is the current $\alpha \longrightarrow \int_Z \alpha$. **It is closed and positive** (Lelong).

Remark: (Poincare-Lelong formula) $\frac{\sqrt{-1}}{\pi} dd^c \log|\varphi| = [Z_\varphi]$, where Z_φ is a divisor of a holomorphic function φ.

Definition: A locally integrable function $f: M \longrightarrow [\infty, \infty[$ is called **plurisubharmonic** (psh) if $dd^c f$ is a positive current.

Claim: (a local dd^c-lemma) **Locally, every positive, closed (1,1)-current is obtained as $dd^c f$,** for some psh function f.

Definition: Let f be a real locally integrable function on a complex manifold, such that $dd^c f + \alpha$ is a positive current, for some smooth $(1,1)$-form α. Then f is called **almost plurisubharmonic**.

Definition: Let L be a line bundle and h a smooth Hermitian metric on L. For any almost plurisubharmonic function f, we call he^{-f} **a singular metric** on L. Its curvature is equal to $\Theta_h + dd^c f$.

Lelong numbers and multiplier ideals

Definition: Let f be an almost plurisubharmonic function, and e^{-f} the corresponding singular metric on a trivial line bundle $\mathcal{O}_M$. **The multiplier ideal** of f is a sheaf of L^2-integrable holomorphic sections of $\mathcal{O}_M$.

THEOREM: (Nadel) **It is a coherent sheaf.**

Remark: The multiplier ideal of f is determined uniquely by the corresponding current $dd^c f$.

Definition: **A Lelong number** $\nu_x(\eta)$ of a closed, positive $(1,1)$-current η at $x \in M$ is defined as a supremum of all $\lambda > 0$ such that $e^{-2\lambda\varphi}$ is integrable in a neighbourhood of x, for some $\eta = dd^c\varphi$.

Remark: $e^{-2\varphi}$ **is integrable in x if and only if the multiplier ideal of φ is trivial in x.**

Definition: For a positive number $c > 0$, **the Lelong set** F_c of a (1,1)-current η is a set of all points $x \in M$ with $\nu(\eta, x) \geqslant c$. From its definition, it is immediate that F_c **is the support for the coherent sheaf** $\mathcal{O}_M/\mathcal{I}(c^{-1}\eta)$**,** hence **the Lelong sets are complex analytic** (Siu, 1974).

Regularized maximum

Claim: Let $\mu : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function, monotonous in all arguments and convex, and $\varphi_1, ..., \varphi_n$ a set of plurisubfarmonic functions. **Then** $\mu(\varphi_1, ..., \varphi_n)$ **is also plurisubharmonic.**

Definition: (Demailly) Let $\mu : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth, convex function, increasing in both arguments. Suppose that for all $|x - y| \geqslant \varepsilon$, one has $\mu(x, y) = \max(x, y)$, and also $\mu(x, y) = \mu(y, x)$, $\mu(y + \alpha, x + \alpha) = \mu(x, y)$. Then μ is called **a regularized maximum** and denoted as $\max_\varepsilon(x, y)$.

Remark: A regularized maximum of smooth plurisubharmonic functions is smooth and psh.

Definition: A nef current is a weak limit of closed, positive forms.

Remark: Let $x \in M$ be a point on a Kähler manifold, and dist_x the corresponding distance function. It is easy to see that **around** x**,** $dd^c \log \operatorname{dist}_x$ **is plurisubharmonic.** Since $\log \operatorname{dist}_x = \lim_{C \longrightarrow -\infty} \max_\varepsilon(\log \operatorname{dist}_x, C)$, $dd^c \log \operatorname{dist}_x$ **is a nef current.**

Lelong numbers for (p, p)-currents

Definition: Let α be a positive, closed current, and η a nef current, $\eta = \lim \eta_i$, with η_i smooth, positive and closed. Define the product $\alpha \wedge \eta := \lim \alpha \wedge \eta_i$. **This limit exists by compactness, is closed and positive.**

A caution: The limit may be non-unique.

Definition: (Demailly) Choose $\eta = dd^c \log \operatorname{dist}_x$ and η_i its approximation constructed using the regularized maximum. For a closed, positive (p, p)-current Θ, define **the Lelong number $\nu_x(\Theta)$** as a mass of a measure $\Theta \wedge (dd^c \log \operatorname{dist}_x)^{n-p}$ carried at x.

Remark: **The Lelong sets are complex analytic** (Siu, 1974).

Siu's decomposition formula: Let Θ be a positive (p, p)-current, and Z_i the p-dimensional components of its Lelong sets, with Lelong numbers c_i (at generic point). **Then $\Theta = \sum_i c_i [Z_i] + R$, where R is closed, positive, and all Lelong sets or R are less than p-dimensional.**

Algebraic multiplier ideals

Let $Z_1, ... Z_n \subset M$ be a set of irreducible subvarieties, and $c_1, ..., c_n$ positive real numbers. Consider a blow-up $M_1 \xrightarrow{p} M$ with simple normal crossings, such that a proper preimage D_i of each Z_i is a divisor.

Definition: **An algebraic multiplier ideal** $\mathcal{I}$ associated with Z_i, c_i is $p_*\left(K_{M_1/M} \otimes \mathcal{O}(\sum_i [c_i D_i])\right)$, where $[c_i D_i]$ is the integer part of the real divisor $c_i D_i$ (rounded down).

Remark: It is independent from the choice of resolution.

Remark: Since $p_*(K_{M_1/M}) = O_M$, for c_i very small, $\mathcal{I} = \mathcal{O}_M$

Claim: Let φ be an almost psh function on M_1 with $dd^c\varphi = \alpha + \sum [c_i D_i]$, where α is smooth, and $\pi_* \varphi$ its pushforward. **Then $\mathcal{I}$ is the multiplier ideal associated with $\pi_* \varphi$.**

Proof: For any weight φ, $\pi_* \mathcal{I}(\varphi) \otimes K_{M_1} = \mathcal{I}(\varphi) \otimes K_M$, hence it suffices to check that $\mathcal{I}(\varphi) = \mathcal{O}(\sum_i [c_i D_i])$. This is a 1-dimensional computation.■

Real b-divisors

Definition: Let M be a complex manifold. A **b-divisor** on M is a choice of a divisor D_{M_1} on each blow-up $M_1 \longrightarrow M$, defined in such a way that $p_*(D_{M_2}) = D_{M_1}$, for any sequence of blow-ups $M_2 \longrightarrow M_1 \longrightarrow M$.

Definition: A b-divisor is called **finite** if there is a blow-up $M_1 \longrightarrow M$ such that for all blow-ups $M_2 \longrightarrow M_1 \longrightarrow M$, D_{M_2} is a proper pre-image of D_{M_1}.

Remark: One can define **real b-divisors** and their integer parts $[D]$ as usual.

Definition: We call a b-divisor D **admissible** if $[kD]$ is finite for all $k > 0$.

Remark: Given a finite real b-divisor D on $M_1 \xrightarrow{p} M$, we define a multiplier ideal of D as $p_*\left(K_{M_1/M} \otimes \mathcal{O}([D])\right)$. For any admissible b-divisor D, the multiplier ideal $\mathcal{I}(kD)$ is well defined for all $k > 0$.

Demailly's regularization and multiplier ideals

Remark: A b-divisor on a blow-up gives a valuation on the field of rational functions on M, with its center the image of this divisor. A b-divisor is uniquely determined by its image in M, which is a formal sum of irreducible subvarieties. The corresponding multiplier ideal is the one defined above.

THEOREM: (Demailly's regularization of positive (1,1)-currents) Let η be a positive, closed (1,1)-current, and $\mathcal{I}(\eta)$ the corresponding multiplier ideal. **Then there exists an admissible b-divisor D such that $\mathcal{I}(k\eta) = \mathcal{I}(kD)$.** The corresponding centers are the Lelong sets, and their coefficients are the Lelong numbers.

THEOREM: (Nadel's vanishing) Let (M, ω) be a Kaehler manifold, η a closed, integer current, $\eta > \varepsilon\omega$, and L a holomorphic line bundle with $[c_1(L) = [\eta]$. Consider a singular metric on L associated with η, and let $\mathcal{I}(L)$ be the sheaf of L^2-integrable sections. **Then $H^i(\mathcal{I}(L) \otimes K_M) = 0$ for all $i > 0$.**

Nef classes and positive currents

Definition: A cohomology class on a Kaehler manifold is called **nef** if it belongs to a closure of a Kähler cone.

Remark: Let α be a positive, closed (1,1)-form (not necessarily positive definite). Clearly, $\alpha + \varepsilon\omega$ is Kähler. **Therefore, the cohomology class of α is nef.**

Remark: Converse is not necessarily true: **there are nef classes which cannot be represented by semipositive forms** (Demailly, Peternell, Schneider).

Claim: Every nef class can be represented by a positive nef current (immediately follows from weak compactness).

Nef classes and positive currents

Claim: Let η be a nef current, and Z a p-dimensional irreducible component of its Lelong set F_c. Denote by $[Z]$ its integration current. **Then $[Z]$ is dominated by η, that is, $\eta^p - c^p[Z]$ is positive.**

Proof. Step 1: Siu's decomposition formula gives $\eta^p = \sum_i c_i[Z_i] + R$, where c_i are Lelong numbers of η^p. To prove the claim, we need $\nu_x(\eta^p) \geqslant \nu_x(\eta)^p$.

Step 2: Slicing and using regularization, we reduce the problem to the case when $p = n$ and η has a single logarithmic pole. Here the inequality is implied by the analytic definition of Lelong numbers.■

Fujiki's formula

THEOREM: (V.) Let M be a simple hyperkaehler manifold, $\dim_{\mathbb{C}} M = 2n$ and $H^*_r(M)$ the part of cohomology generated by $H^2(M)$. **Then $H^*_r(M)$ is isomorphic to the symmetric algebra (up to the middle degree).**

Remark: The multiplication in $H^*_r(M)$ is $SO(H^2(M), q)$-invariant. Not many ways to write multiplication invariantly.

THEOREM: (Fujiki's formula)
Let $\eta_1, ..., \eta_{2n} \in H^2(M)$ be cohomology classes. Then

$$\eta_1 \wedge \eta_2 \wedge ... = const \sum_\sigma q(\eta_{\sigma_1} \eta_{\sigma_2}) q(\eta_{\sigma_3} \eta_{\sigma_3}) q(\eta_{\sigma_{2n-1}} \eta_{\sigma_{2n}})$$

Parabolic nef classes on hyperkaehler manifolds

Definition: A real cohomology class $[\eta] \in H^{1,1}(M)$ on a simple hyperkähler manifold is called **parabolic** if $q([\eta],[\eta]) = 0$, that is, $\int_M \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} = 0$.

Definition: Let M be a simple hyperkaehler manifold, η a nef current representing a parabolic class $[\eta] \in H^{1,1}(M)$. We say that a subvariety $Z \subset M$ is **$[\eta]$-coisotropic** if η dominates the current of integration $[Z]$.

Remark: We have proved that **all Lelong sets of η are $[\eta]$-coisotropic.**

Definition: Let (M, Ω) be a a holomorphic symplectic manifold, $\dim_{\mathbb{C}} Z = 2n$, and $Z \subset M$ a complex subvariety of codimension $p \leqslant n$. Then Z is called **coisotropic** if the restriction $\Omega^{n-p+1}|_Z$ vanishes on all smooth points of Z.

Remark: This is equivalent to Ω having rank $n-p$ on TZ.

$[\eta]$-coisotropic subvarieties in hyperkaehler manifolds

THEOREM: Let M be a simple hyperkaehler manifold, $\dim_{\mathbb{C}} M = 2n$ and $[\eta]$ a parabolic nef class. **Then all $[\eta]$-coisotropic subvarieties are coisotropic.**

Proof. Step 1: $[\eta]^{n+1} = 0$ (Fujiki's formula). A positive, closed current which is cohomologous to zero vanishes. Therefore, $\eta^{n+1} = 0$. This implies that $\dim Z \geqslant n$.

Proof. Step 2: Since $\eta^p - c[Z]$ is positive, $\eta^{n+1} - c[Z] \wedge \eta^{n-p+1}$ is also positive. Therefore, $[Z] \wedge \eta^{n-p+1} = 0$.

Proof. Step 3: The form $\Omega^i \wedge \overline{\Omega}^i$ is (weakly) positive. Therefore, $\eta^p \wedge \Omega^{n-p+1} \wedge \overline{\Omega}^{n-p+1}$ is positive. This form is cohomologous to 0 (Fujiki's formula). This gives $\eta^p \wedge \Omega^{n-p+1} \wedge \overline{\Omega}^{n-p+1} = 0$.

Proof. Step 4:

$$0 = \eta^p \wedge \Omega^{n-p+1} \wedge \overline{\Omega}^{n-p+1} \geqslant [Z] \wedge \Omega^{n-p+1} \wedge \overline{\Omega}^{n-p+1} = 0$$

■

Lelong numbers of nef classes

Corollary: Let η be a nef class on a generic hyperkähler manifold M . **Then all Lelong numbers of η vanish.**

Proof: Indeed, all subvarieties of M are symplectic.