Hyperkähler SYZ conjecture and multiplier ideal sheaves

Misha Verbitsky

HOLOMORPHICALLY SYMPLECTIC VARIETIES AND MODULI SPACES
June 5, 2009, Lille

Lecture 3

Hyperkähler manifolds.

Definition: **A hyperkähler manifold** is a compact, Kähler, holomorphically symplectic manifold.

Definition: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = q(\eta,\eta)^n$, for some integer quadratic form q on $H^2(M)$.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by this relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$C_{2n-2}^{n-1} q(\eta,\eta) = (n/2) \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - (1-n) \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right) \left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π is conjectured to be rational. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be used to determine the topology of M (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about M, it's reasonable to start from a holomorphic Lagrangian fibration (if it exists).

The SYZ conjecture

DEFINITION: Let (M, ω) be a Calabi-Yau manifold, Ω the holomorphic volume form, and $Z \subset M$ a real analytic subvariety, Lagrangian with respect to ω. If $\Omega|_Z$ is proportional to the Riemannian volume form, Z is called **special Lagrangian** (SpLag).

(Harvey-Lawson): **SpLag subvarieties minimize Riemannian volume in their cohomology class.** This implies that their moduli are finite-dimensional.

A trivial remark: A holomorphic Lagrangian subvariety of a hyperkähler manifold (M, I) is special Lagrangian on (M, J), where (I, J, K) is a quaternionic structure associated with the hyperkähler structure.

Another trivial remark: A smooth fiber of a Lagrangian fibration has trivial tangent bundle. In particular, **a smooth fiber of a holomorphic Lagrangian fibration is a torus.**

Strominger-Yau-Zaslow, "Mirror symmetry as T-duality" (1997). Any Calabi-Yau manifold admits a Lagrangian fibration with special Lagrangian fibers. Taking its dual fibration, one obtains "the mirror dual" Calabi-Yau manifold.

Nef classes and semiample bundles

DEFINITION: A cohomology class η is called **nef** if it belongs to the closure of the Kähler cone. A holomorphic line bundle L is **nef** if $c_1(L)$ is nef.

DEFINITION: A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

REMARK: **From semiampleness it obviously follows that L is nef.** Indeed, let $\pi : M \longrightarrow \mathbb{P}H^0(L^N)^*$ the the standard map. Since sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of L is a pullback of the Kähler form on $\mathbb{C}P^n$.

REMARK: The converse is false:
a nef bundle is not necessarily semiample.

The hyperkähler SYZ conjecture

CONJECTURE: (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

REMARK: This is the only known source of SpLag fibrations.

Definition: A nef class η is called **parabolic** if $q(\eta, \eta) = 0$.

A trivial observation: Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^* \omega_X$ is nef and parabolic.**

The hyperkähler SYZ conjecture: Let L be a parabolic nef line bundle on a hyperkähler manifold. Then L is semiample.

Recollection (lecture 1). Generic manifolds.

Definition: Let (M, I) be a Kähler manifold, and $\mathcal{I} \in \operatorname{End}(H^*(M, \mathbb{R}))$ **the Hodge decomposition operator** acting on (p, q)-forms as a multiplication by $(p-q)\sqrt{-1}$. Consider the smallest rational Lie subalgebra containing $\mathcal{I}$, and let $G_{MT} \subset GL(H^*(M, \mathbb{R})$ be the corresponding Lie group. It is called **the Mumford-Tate group of (M, I)**,

Remark: (Lower semicontinuity of Mumford-Tate group) Let S be a holomorphic family of complex structures of Kaehler type on a compact manifold M. Then $G_{MT}(M, I)$ **is the same for all I outside of a union Z of proper Zariski closed subsets**, and, moreover, $G_{MT}(I') \subset G_{MT}(M, I)$ for all $I' \in S$.

Definition: Complex structures outside of Z are called **Mumford-Tate generic.**

THEOREM: Let I be a Mumford-Tate generic complex structure on a simple hyperkaehler manifold, and $Z \subset (M, I)$ a complex subvariety. Then Z **is symplectic outside of its singularities.** Moreover, a normalization $\tilde{Z}$ of Z is holomorphically symplectic.

Recollection (Lecture 1). The Kähler cone.

Theorem: Let M be a simple hyperkähler manifold such that all integer $(1,1)$-classes satisfy $q(\nu,\nu) \geqslant 0$. **Then its Kähler cone is one of two components K_+ of a set $K := \{\nu \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ q(\nu,\nu) > 0\}$.**

Corollary: Let $\eta \in H^{1,1}(M)$, $q(\eta,\eta) \geqslant 0$ be an integer class in a simple hyperkähler manifold with $NS(M) = \mathbb{Z}$. **Then η is nef** .

Remark: **Such classes are easy to construct** using surjectivity of the period map.

Recollection (Lecture 2). Positive currents.

Definition: **A current** on a manifold is a differential form with coefficients in generalized functions.

Definition: **A cone of strongly positive (p,p)-forms** is a closed, convex cone generated by $\eta_1 \wedge \eta_2 \wedge ... \wedge \eta_p$, for all semipositive Hermitian (1,1)-forms $\eta_1, ..., \eta_p$.

Definition: A **weakly positive current** is a current Θ which satisfies $\int_M \Theta \wedge \alpha \geqslant 0$ for any positive form with compact support.

Theorem: **The space of positive currents with $\int_M \eta \wedge \omega^{n-p} \leqslant C$ is compact.**

Corollary: **A nef cohomology class is represented by a positive, closed current.**

Recollection (Lecture 2): Lelong numbers and multiplier ideals

Definition: Let h be a singular metric on a holomorphic line bundle L, with its curvature Θ a positive, closed current. The corresponding **multiplier sheaf** $\mathcal{I}(\eta)$ is a sheaf of L^2-integrable sections of L.

THEOREM: (Nadel) **It is a coherent sheaf.**

Definition: **A Lelong number** $\nu_x(\eta)$ of a closed, positive $(1,1)$-current η at $x \in M$ is defined as a supremum of all $\lambda > 0$ such that $\mathcal{I}(\lambda\eta) = \mathcal{O}_M$ in a neighbourhood of x.

Definition: For a positive number $c > 0$, **the Lelong set** F_c is a support of the coherent sheaf $\mathcal{O}_M/\mathcal{I}(c^{-1}\eta)$.

THEOREM: (Nadel's vanishing) Let (M, ω) be a Kaehler manifold, η a closed, integer current, $\eta > \varepsilon\omega$, and L a holomorphic line bundle with $[c_1(L) = [\eta]$. Consider a singular metric on L associated with η, and let $\mathcal{I}(L)$ be the sheaf of L^2-integrable sections. **Then $H^i(\mathcal{I}(L) \otimes K_M) = 0$ for all $i > 0$.**

Recollection (Lecture 2): Lelong sets for parabolic nef classes.

THEOREM: Any component of a Lelong set of a parabolic nef class on a hyperkähler manifold is coisotropic.

THEOREM: For a generic hyperkähler manifold, all Lelong sets of parabolic nef classes are empty.

Remark: Fujiki's formula was used in a proof. Let $\eta_1, ..., \eta_{2n} \in H^2(M)$. Then

$$\eta_1 \wedge \eta_2 \wedge ... = const \sum_\sigma q(\eta_{\sigma_1}\eta_{\sigma_2})q(\eta_{\sigma_3}\eta_{\sigma_3})q(\eta_{\sigma_{2n-1}}\eta_{\sigma_{2n}})$$

(the sum is taken over all permutations $\sigma \in S_{2n}$.)

Semipositive line bundles

MAIN THEOREM: Let L be a parabolic line bundle on a hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$ with all Lelong sets F_C empty for $c > 2^{-2^{2n}-1}$. Then L is $\mathbb{Q}$-effective, for some $k > 0$.

Remark: This implies that **any simple hk manifold with a parabolic nef class admits a coisotropic subvariety.**

Plan of a proof:

Step 1. Show that $H^*(L^N)$ is non-zero, for all N.
For $N < 2^{2^{2n}+1}$, one has $L^N = \mathcal{I}(L^N)$.

Step 2. Construct an embedding

$$H^i(\mathcal{I}(L^N)) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes \mathcal{I}(L^N)).$$

Step 3. THEOREM: Let L be a nef bundle on a hyperkähler manifold, with $q(L, L) = 0$. Assume that $H^0(\Omega^*(M) \otimes L^N) \neq 0$, for all $0 < N < 2^{2^{2n}+1}$.
Then L^k is effective, for some $k > 0$.

Step 1. Show that $H^*(L^N)$ is non-zero, for all N.

This is actually clear, because $\chi(L) = P(q(L,L))$, where P is a polynomial with coefficients depending on Chern classes of M only (Fujiki). Then

$$\chi(L) = \chi(\mathcal{O}_M) = n + 1$$

(Bochner's vanishing).

Step 2. Construct an embedding

$$H^i(\mathcal{I}(L^N) \otimes K) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes \mathcal{I}(L^N)).$$

This is called **"Hard Lefschetz theorem with coefficients in L"** (Demailly-Peternell-Schneider).

Idea of a proof: The proof is the same as Nadel's vanishing theorem. Let $B := L^*$. Then

$$\Delta_{\nabla'} - \Delta_{\overline{\partial}} = [\Theta_B, \Lambda] \leqslant 0,$$

therefore $H^i(B^N) = \ker \Delta_{\overline{\partial}} \subset \ker \Delta_{\nabla'}$, and the last space is identified with B^*-valued holomorphic differential forms.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \operatorname{Vol}_M$$

Kobayashi-Hitchin correspondence (part 2)

Kobayashi-Hitchin correspondence (Donaldson, Uhlenbeck-Yau) Let B be a holomorphic vector bundle. **Then B admits Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

EXAMPLE: Let M be a Kähler-Einstein manifold. Then TM is polystable.

REMARK: Let M be a Calabi-Yau (e.g., hyperkähler) manifold. Then TM admits a Hermitian-Einstein metric for any Kähler class (Calabi-Yau theorem). **Therefore, TM is stable for all Kähler structures.**

Recollection (Lecture 1): The modified nef cone.

Definition: A class $\eta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive current.

The divisorial Zariski decomposition theorem: (S. Boucksom) Let M be a simple hyperkähler manifold. Then **every pseudoeffective class can be decomposed as a sum** $\eta = \nu + \sum_i a_i E_i$, where ν is nef, a_i positive numbers, and E_i exceptional divisors satisfying $q(E_i, E_i) < 0$.

Definition: A **modified nef cone** (also "birational nef cone" and "movable nef cone") is a closure of a union of all nef cones for all bimeromorphic models of a holomorphically symplectic manifold M.

THEOREM: (D. Huybrechts, S. Boucksom)
The modified nef cone is dual to the pseudoeffective cone under the Bogomolov-Beauville-Fujiki pairing.

Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$

THEOREM 1: Let M be a compact hyperkähler manifold, $\mathfrak{T}$ a tensor power of a tangent bundle (such as a bundle of holomorphic forms), and $E \subset \mathfrak{T}$ a coherent subsheaf of $\mathfrak{T}$. Then the class $-c_1(E) \in H^{1,1}_{\mathbb{R}}(M)$ is pseudoeffective.

Proof. Step 1: Polystability implies that $\int_M c_1(E) \wedge \omega^{n-1} \leqslant 0$ for any Kähler class ω. Equivalently, $q(c_1(E), \omega) \leqslant 0$ (Fujiki's formula). This means that **the class $-c_1(E)$ lies in the dual nef cone**.

Step 2: Let $M_\alpha \xrightarrow{\varphi} M$ be a hyperkähler manifold birationally equivalent to M. Then φ is non-singular in codimension 1. Let $\mathfrak{T}_\alpha$ be the same tensor power of TM_α as $\mathfrak{T}$. Clearly, $\mathfrak{T}_\alpha$ can be obtained as a saturation of $\varphi^*\mathfrak{T}$. **Taking a saturation of $\varphi^* E \subset \varphi^* \mathfrak{T}$, we obtain a coherent subsheaf $E_\alpha \subset \mathfrak{T}_\alpha$, with $c_1(E_\alpha) = c_1(E)$.**

Step 3: We obtained that **the class $-c_1(E)$ lies in the dual nef cone of M_α, for all birational models of M.** Now apply Huybrechts-Boucksom. ■

Remark: Another proof was obtained earlier by Campana-Peternell (for projective manifolds).

L-valued holomorphic forms are non-singular in codimension 1

THEOREM 2: Let M be a compact hyperkähler manifold, L a nef line bundle satisfying $q(L,L)=0$, $\mathfrak{T}$ some tensor power of a tangent bundle, and $\gamma \in H^0(\mathfrak{T}\otimes L)$. **Assume L is not $\mathbb{Q}$-effective. Then γ is non-singular in codimension 1.**

Proof: Follows from the previous theorem.

Remark: Also known to Campana-Peternell.

From L-valued differential forms to sections of L

THEOREM 3: Let L be a nef bundle on a hyperkähler manifold, with $q(L,L) = 0$. Assume that $H^0(\Omega^*(M) \otimes L^N) \neq 0$, for all $N < 2^{2^{2n}+1}$. **Then L^k is effective, for some $k > 0$.**

Definition: Let S be a coherent sheaf. For a set of sections $s_1, s_2, ..., s_k$, $s_i \in S \otimes L^{k_i}$, denote by $\langle s_1, s_2, ..., s_k \rangle \subset S$ the saturation of a sheaf generated by rank 1 subsheaves $s_i \otimes L^{-k_i} \subset S$. We say that $s_1, ..., s_k$ are **linearly independent** if $\operatorname{rk}\langle s_1, s_2, ..., s_k \rangle = k$.

Remark: If $f_1, ..., f_k$ and $g_1, ..., g_k$ are independent, $f_i \in L^{-k_i}$, $g_i \in L^{-l_i}$, and $\langle f_1, f_2, ..., f_k \rangle = \langle g_1, g_2, ..., g_k \rangle$, **the fraction $\frac{f_1 \wedge f_2 \wedge ...}{g_1 \wedge g_2 \wedge ...}$ is a section of $L^{\sum k_i - \sum l_i}$.**

Remark: If $S = \mathfrak{T}$ is a tensor bundle on a hk manifold, and L is parabolic nef and not $\mathbb{Q}$-effective, then all sections and non-zero in codimension 1, hence **the fraction $\frac{f_1 \wedge f_2 \wedge ...}{g_1 \wedge g_2 \wedge ...}$ gives a trivialization of $L^{\sum k_i - \sum l_i}$.**

Proof of Theorem 3.

Step 1: Let $i_k := 2^k$, $k = 0, 2, ..., 2^{2n}$. For each i_p, choose $f_p \in \Omega^* M \otimes L^{i_p}$. **Since** $\dim \Omega^* M = 2^n$**, some of these sections are linearly dependent.**

Step 2: Choose distinct k-tuples $f_{p_1}, f_{p_2}, ... f_{p_k}$, $f_{q_1}, f_{q_2}, ... f_{q_k}$, linearly independent, and satisfying $\langle f_{p_1}, f_{p_2}, ... f_{p_k} \rangle = \langle f_{q_1}, f_{q_2}, ... f_{q_k} \rangle$.

Step 3: By previous remark, unless L is $\mathbb{Q}$-effective, $\frac{f_{p_1} \wedge f_{p_2} \wedge ...}{f_{q_1} \wedge f_{q_2} \wedge ...}$ **gives a trivialization of** $L^{\sum i_{p_i} - \sum i_{q_i}}$.

Step 4: Then $\sum i_{p_i} - \sum i_{q_i}$, hence the k-tuples $f_{p_1}, f_{p_2}, ... f_{p_k}$, $f_{q_1}, f_{q_2}, ... f_{q_k}$ coincide. Indeed, **a binary representation of an integer with 1 at position** p_i **is equal to** $\sum i_{p_i}$. ■