Multiple fibers of holomorphic Lagrangian fibrations

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Hyperkähler manifolds of maximal holonomy

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

Further on, **all hyperkähler manifolds are assumed to be of maximal holonomy**.

Lagrangian fibrations

THEOREM: (Matsushita) Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a surjective holomorphic map, with $0 < \dim X < \dim M$. **Then π is a Lagrangian fibration** (that is, has holomorphic Lagrangian fibers).

THEOREM: (Hwang) In these assumptions, **X is biholomorphic to $\mathbb{C}P^n$ when it is smooth.**

CONJECTURE: X is biholomorphic to $\mathbb{C}P^n$ **when it is normal.**

THEOREM: (Matsushita)
Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a Lagrangian fibration, with X normal. **Then $H^*(X,\mathbb{Q}) \cong H^*(\mathbb{C}P^n,\mathbb{Q})$.**

REMARK: General fibers of π are Abelian varieties (projective complex tori), by Arnold-Liouville. Conversely, as shown by Hwang-Weiss, **any Lagrangian complex torus in M is a fiber of a Lagrangian fibration.**

Multiplicity of an irreducible component of a fiber of complex fibration

REMARK: **A fibration** $\pi: M \to X$ is a proper surjective holomorphic map of complex manifolds. **We shall always tacitly assume** that the fibration is proper and **equidimensional**, that is, the irreducible components of all fibers have dimension $\dim M - \dim X$. **A special fiber** of π is a fiber containing a critical point

DEFINITION: Let Z be an irreducible component of a special fiber of a fibration $\pi: M \to X$, and $z = \pi(Z)$. **The multiplicity** of Z is the rank of $\pi^*(\mathcal{O}_X/\mathfrak{m}_z)$ in a general point of Z, where $\mathfrak{m}_z \subset \mathcal{O}_X$ is the maximal ideal of Z.

REMARK: Suppose that $\dim X = 1$. Locally in M, around a general point of $m \in Z$, there is a coordinate system $x_1, ..., x_n$, such that $\pi(x_1, ..., x_n) = x_1^d$. **In this case, d is multiplicity of Z.**

REMARK: Another definition of multiplicity is more geometric, but it is equivalent to the one given above; **the equivalence is left as an exercise.**

DEFINITION: Let Z be an irreducible component of a special fiber of a fibration $\pi: M \to X$, and $z = \pi(Z)$. Consider a small disc $D \subset M$ of dimension $\dim X$ transversal to Z and intersecting it in m. **The multiplicity** of Z in $m \in M$ is the number of intersection points between D and a general fiber of π which is sufficiently close to Z.

Multiplicity of a fiber

THEOREM: (Campana-Kamenova-V.)
Let $\pi: M \longrightarrow X$ be an abelian fibration (that is, a fibration with general fiber a complex torus), and assume that M is Kähler. Let μ_i be the multiplicity of irreducible components pf $\pi^{-1}(z)$ for some $z \in X$, and $gcd(\mu_i)$ their greatest common divisor. **Then $gcd(\mu_i) = \min \mu_i$.**

DEFINITION: Let $\pi: M \longrightarrow X$ be a surjective holomorphic map of complex manifolds, and $D \subset X$ its set of critical values, which is known as **the discriminant**, or **the discriminant divisor** (it has codimension 1). We say that π **has no multiple fibers in codimension 1** if for a general point $x \in D$, the fiber $\pi^{-1}(x)$ has a component with multiplicity 1.

THEOREM: Let $\pi: M \to X$ be an elliptic fibration on a K3 surface. **Then π has no multiple fibers.**

The main result today is a generalization of this theorem

MAIN THEOREM: Let $\pi: M \to X$ be a Lagrangian fibration on a hyperkähler manifold. **Then π no multiple fibers in codimension 1.**

Second cohomology of a hyperkähler manifold is torsion-free

CLAIM: Let M be a hyperkähler manifold of maximal holonomy. **Then $H^2(M)$ is torsion-free.**

Proof: The universal coefficients formula gives the exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $H_1(X,\mathbb{Z}) = 0$ for a maximal holonomy hyperkähler manifold, **this gives an isomorphism $H^2(X;\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z})$, hence the torsion vanishes.** ■

Primitive classes and Lagrangian fibrations

Recall that a class $\eta \in H_k(M,\mathbb{Z})$ is called **primitive** if it is not divisible, that is, there is no $\eta' \in H_k(M,\mathbb{Z})$ such that $\eta = r\eta'$, with $r \in \mathbb{Z}$, $|r| \geqslant 2$.

THEOREM 1: Let $\pi: \ M \longrightarrow \mathbb{C}P^n$ be a Lagrangian fibration on a hyperkähler manifold of maximal holonomy, and $H \subset \mathbb{C}P^n$ a hyperplane section. **Then the following assertions are equivalent.**

(i) The homology class of $\pi^{-1}(H)$ is primitive.

(ii) The map π has has no multiple fibers in codimension 1.

Proof. Step 1: The implication (ii) $\Rightarrow$ (i) is proven later today using the ETMDPS vanishing theorem. The converse implication: let D_1 be an irreducible component of the discriminant D; assume it has multiplicity μ. **We need to show that the preimage of the hyperplane section H is not primitive.**

Step 2: Arguing ad absurdum, assume that D_1 is homologous to $k[H]$, but $[\pi^{-1}(H)]$ is primitive in $H^2(M,\mathbb{Z})$. Since $[D_1] = k[H]$, $\pi^{-1}(D_1)$ is the zero set of a section $s \in H^0(\mathbb{C}P^n, \mathcal{O}(k))$. The pullback of this section has multiplicity μ; since $\pi^*(\mathcal{O}(1))$ is primitive, k is divisible by μ. The universal coefficients formula implies that $H^2(M,\mathbb{Z})$ is torsion-free. Therefore, $\pi^* s = s_0^\mu$, where $s_0 \in H^0(M, \pi^*\mathcal{O}(k/\mu))$. Since $\pi^*\mathcal{O}(k/\mu)$ is trivial on fibers of π, the section s_0 is a pullback of a section $s_1 \in H^0(\mathbb{C}P^n, \mathcal{O}(k/\mu))$. **This is impossible, because s and hence s_1 vanishes on D_1, which has degree k.** ■

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{2n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Hirzebruch-Riemann-Roch formula

DEFINITION: Let B be a holomorphic vector bundle (or a coherent sheaf). The **holomorphic Euler characteristic** is $\chi(L) := \sum_i (-1)^i H^i(M,B)$.

THEOREM: (Riemann-Roch-Hirzebruch) Let M be a compact complex manifold, and B a holomorphic vector bundle. **Theb $\chi(B)$ can be expressed through the the Chern classes of TM and B,** $\chi(B) = \int_M td(TM) \wedge ch(B)$ where td is the the Todd polynomial on Chern classes of TM,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

and $ch(B)$ its Chern character,

$$ch(B) = 1 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + ...$$

■

Hirzebruch-Riemann-Roch formula and BBF form

THEOREM: (Huybrechts) Let M be a hyperkähler manifold, $\dim_C M = 2n$ and L a holomorphic line bundle. **Then $\chi(L) = \sum a_i q(c_1(L))^i$, where the coefficients a_i are constants depending on the topology of M.**

Proof. Step 1: Let A^* be the subalgebra in cohomology generated by $H^2(M)$. **Then $A^{2i} \cong \operatorname{Sym}^i(H^2(M))$ up to the middle degree, and $A^{n+i} \cong \operatorname{Sym}^{n-i}(H^2(M))$; there is an $O(H^2(M))$-action on cohomology, and the multiplication is $O(H^2(M))$-invariant (V., 1995).**

Step 2: All Chern classes of TM are $O(H^2(M))$-invariant, but there is only one (up to a constant multiplier) $O(H^2(M))$-invariant functional on $\operatorname{Sym}^{2i}(H^2(M))$. On the class $\eta^{2i} \in H^{4i}(M)$ this functional takes value $q(\eta,\eta)^i$. Therefore, **all L-dependent coefficients in the Hirzebruch-Riemann-Roch formula for $\chi(L)$ are expressed through $q(c_1(L))$.** ∎

COROLLARY: Let L be a line bundle on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$. Assume that $q(c_1(L)) = 0$. **Then $\chi(L) = n + 1$.**

Proof: Indeed, $\chi(L) = \chi(\mathcal{O}_M) = n + 1$, with the second equality implied by Bochner's vanishing theorem. ∎

ETMDPS vanishing theorem

DEFINITION: A real $(1,1)$-form η on a complex manifold M is called **semi-positive** if $\eta(x, Ix) \geqslant 0$ for all real tangent vectors x.

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we do not use) is due to Demailly, Peternell and Schneider.

THEOREM: Let (M, I, ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, K its canonical bundle, and L a holomorphic line bundle on M equipped with a Hermitian metric h. Assume that the curvature Θ of L is a positive form on M. Then **the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map**

$$H^0(\Omega^{n-i} M \otimes L) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L).$$

Here ω is considered as an element in $H^1(\Omega^1 M)$, and multiplication by ω maps $H^k(\Omega^{n-l} M \otimes L)$ to $H^{k+1}(\Omega^{n-l+1} M \otimes L)$.

Primitivity and vanishing of cohomology

Lemma 2: Let B be a vector bundle equipped with a filtration $0 = B_0 \subset B_1 \subset ... \subset B_k = B$. Assume that $H^0(B_i/B_{i-1}) = 0$ for all i. **Then $H^0(B) = 0$.** ■

THEOREM: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L is trivial on all smooth fibers of π.**

Proof. Step 1: Let F be a smooth fiber of π, which is an abelian variety by Arnol'd-Liouville. Then $T^*M|_F$ is an extension of a trivial bundle TF with another trivial bundle $NF = T^*F$. **For any non-trivial line bundle $L \in \mathrm{Pic}_0(F)$, we have $H^0(L \otimes TF) = 0$ and $H^0(L \otimes NF) = 0$, which implies that $H^0(L \otimes T^*M|_F) = 0$.** Similarly one $H^0(L \otimes \Lambda^k T^*M|_F) = 0$ (Lemma 2).

Step 2: Unless L is trivial on F, we have $H^0(L \otimes \Lambda^k T^*M|_F) = 0$, which implies that $H^0(L \otimes \Lambda^* M) = 0$. By Enoki-Mourugane-Takegoshi-Demailly-Peternell-Schneider theorem, **this implies that $H^i(L) = 0$, hence $\chi(L) = 0$, contradicting the formula $\chi(L) = n + 1$.** ■

Fiberwise monodromy of a line bundle

Proposition 1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L admits a connection ∇ which is flat on each restriction $L|_F$ to the fiber of π.**

Proof: Choose a constant metric h^k on $L^{\otimes k}|_F = \mathcal{O}_F$ and let h be its k-th root, which is a metric on $L|_F$. **Since h^k is constant on each fiber, its curvature vanishes, and the Chern connection ∇ associated with h is also flat.**

DEFINITION: **Fiberwise monodromy** of L is its monodromy on the fibers of π.

REMARK: Clearly, **$L = \pi^* L_0$ if and only if the fiberwise monodromy of L on each fiber is trivial.**

REMARK: Since $\mathcal{O}(1)$ is clat on $\mathbb{C}^n = \mathbb{C}P^n \backslash H$, we also obtain a flat connection on $L_0|_{\pi^{-1}(\mathbb{C}P^n \backslash H)}$. **The main theorem would follow if we prove that the monodromy of this flat connection is trivial on $\pi^{-1}(z)$**, where $z \in D$ is a general point of the discriminant.

Stable bundles

DEFINITION: Let F be a torsion-free coherent sheaf F on M. Define **the degree** $\deg_\omega F := \int_M c_1(F) \wedge \omega^{n-1}$, where ω is a Kähler form. Let $\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$. A torsion-free sheaf F is called **stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

THEOREM: On a hyperkähler manifold of maximal holonomy, **the bundle $\Omega^{2i+1}(M)$ is stable for all i, and the bundle $\Omega^{2i}(M)$ is polystable.**

Proof: Follows from the Kobayashi-Hitchin correspondence. ■

REMARK: Clearly, a tensor product of a stable bundle and a line bundle **is also stable.**

Stability and primitive line bundles

COROLLARY: Let $\pi: M \to \mathbb{C}P^n$ be a Lagrangian fibration, and L the primitive bundle constructed above, $L^k = \pi^*(\mathcal{O}(1))$. Let u a section of $L \otimes \Omega^{2i}(M)$. **Then u is non-zero somewhere on $\pi^{-1}(D)$, where D is the discriminant of π.**

Proof: Suppose that $u = 0$ on D. Then the degree if the line sub-bundle V of $L \otimes \Omega^{2i}(M)$ generated by u is at least $\deg \pi^{-1}(D) = \deg_{\mathbb{C}P^n} D \cdot \deg_\omega \pi^*(\mathcal{O}(1))$. Then

$$\operatorname{slope}(V) = \deg_{\mathbb{C}P^n} D \cdot \deg_\omega \pi^*(\mathcal{O}(1)) > \\ > \operatorname{slope}(L \otimes \Omega^{2i}(M)) = \deg_\omega L = k^{-1} \deg_\omega \pi^*(\mathcal{O}(1)).$$

This contradicts polystability of $L \otimes \Omega^{2i}(M)$. ■

To prove that π has multiplicity 1 in a general point of the discriminant, **we need to show that L is trivial on a general point of $\pi^{-1}(D)$;** in this case, $L = \pi^*(L_0)$ outside of codimension 2, and Hartogs implies that $L = \pi^*(L_0)$, hence $\mathcal{O}(1)$ is primitive. **The next theorem implies that the monodromy of $L\big|_{\pi^{-1}(z)}$ is trivial for a general $z \in D$, which finishes the proof of primitivity of $\pi^*(\mathcal{O}(1))$.**

Fiberwise monodromy of line bundles

THEOREM: Let $\pi: M \to \mathbb{C}P^n$ be a Lagrangian fibration on a hyperkähler manifold, $L = \pi^*(\mathcal{O}(1))$, and L a primitive line bundle such that $L^{\otimes k} = L$. Consider the fiberwise flat connection ∇ on L constructed above. **Then the monodromy of ∇ is trivial on $\pi^{-1}(z)$**, where $z \in D$ is a general point of the discriminant.

Proof. Step 1: By ETMDPS vanishing theorem, $\dim H^0(\Omega^*(M)\otimes L) \geqslant n+1$. **This implies existence of non-trivial sections of $\Omega^*(M) \otimes L$.**

Step 2: By the previous theorem, we can assume that there is a section $u \in H^0(\Omega^*(M) \otimes L))$ which does not vanish on $\pi^{-1}(z)$.

Step 3: Let $z \in D$ be a general point of the discriminant. Assume that L has non-trivial monodromy on $\pi^{-1}(z)$. Locally around a general point $m \in \pi^{-1}(z)$, the Lagrangian projection has a coordinate form

$$(z_1, ..., z_n, z_{n+1}, ..., z_{2n}) \to (z_1^k, z_2, ..., z_n),$$

with the holomorphic symplectic form expressed as $\sum_{i=1}^n dz_i \wedge dz_{i+n}$. Let $\sigma: X' \to \mathbb{C}P^n$ be a ramified cover of $\mathbb{C}P^n$ ramified in D of ramification index k. In a neighbourhood of the preimage of m, the variety X' has coordinates $\zeta_1, z_2, ..., z_n$, with $\zeta_1^k = z_1$. The corresponding deck transform group $\Gamma = \mathbb{Z}/k$ acts on M' by taking $(\zeta_1, z_2, ..., z_n, a)$ to $(\varepsilon\zeta_1, z_2, ..., z_n, \varphi(a))$, where φ is a fiberwise automorphism of M' of order k, and ε a k-th root of unity.

Fiberwise monodromy of line bundles (2)

Step 4: Let $M' := M \times_{\mathbb{C}P^n} X' \stackrel{\pi'}{\to} X'$ be the fibered product, L' the pullback of L to M', and $z' \in X'$ be a preimage of a general $z \in D$. By construction, the map $M' \to M$ is a covering outside of codimension 2, and the projection $\pi' : M' \to X'$ is Lagrangian and submersive outside of codimension 2. The holomorphically symplectic form can be used to trivialize the sheaf $\Omega^1 {\pi'}^{-1}(z')$ outside in the singular locus of π'. This gives a filtration with trivial subquotients on $\Omega^*(M')$ outside of the singular locus. **Existence of non-zero sections of $\Omega^*(M') \otimes L'$ implies that L' has trivial monodromy on the fibers of π'** (Lemma 2).

Step 5: Arguing ad absurdum, assume that $L\Big|_{\pi^{-1}(z)}$ has non-trivial monodromy. Let $\sigma : M' \to M$ denote the covering defined in Step 4, and $\Gamma = \mathbb{Z}/k$ its deck transform group. Since L' has trivial monodromy, the sections $u \in H^0(\Omega^p(M) \otimes L))$ correspond to sections of $\sigma^* \Omega^p(M')$, on which the deck transform group Γ acts by a k-th root of unity. However, in a neighbourhood of ${\pi'}^{-1}(z')$, the sheaf $\Omega^p(M')$ is an extension of trivial bundles outside of the singular locus of π', and the deck transform action is compatible with this trivialization. Therefore, any section of $\sigma^* \Omega^p(M')$, on which the rotation around the divisor $\zeta_1 = 0$ acts by roots of unity vanishes on this divisor. ■