An intrinsic volume functional on almost complex 6-manifolds and nearly Kähler geometry

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Almost complex manifolds with non-degenerate Nijenhuis tensor

DEFINITION: Let (M,I) be an almost complex manifold. **The Nijenhuis tensor** maps two $(1,0)$-vector fields to the $(0,1)$-part of their commutator.

REMARK: This map is C^∞-linear, and **vanishes precisely when I is integrable** (Newlander-Nirenberg). We write the Nijenhuis tensor as

$$N: \Lambda^2 T^{1,0}(M) \longrightarrow T^{0,1}(M).$$

REMARK: The dual map

$$N^*: \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$$

is also called the Nijenhuis tensor. Cartan's formula implies that N^* acts on $\Lambda^1(M)$ as the $(2,-1)$-part of the de Rham differential.

DEFINITION: Let (M,I) be an almost complex manifold of real dimension 6. We say that N is **everywhere non-degenerate** if $N^*: \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$ is an isomorphism of vector bundles.

Canonical volume on complex manifolds with non-degenerate Nijenhuis tensor

REMARK: The determinant $\det N^*$ gives a section

$$\det N^* \in \Lambda^{3,0}(M)^{\otimes 2} \otimes \Lambda^{0,3}(M)^*.$$

Taking

$$\det N^* \otimes \overline{\det N^*} \in \Lambda^{3,0}(M) \otimes \Lambda^{0,3}(M) = \Lambda^6(M)$$

we obtain a nowhere degenerate real volume form Vol_I on M.

REMARK: This gives a functional $I \xrightarrow{\Psi} \int_M \operatorname{Vol}_I$ on the space of almost complex structures.

We are interested in its extrema, which, as it turns out, correspond to "nearly Kähler almost complex structures".

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: \; B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Classification of holonomies.

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

Holonomy of Riemannian cones

DEFINITION: Let (M,g) be a Riemannian manifold. The **Riemannian cone** of M is $C(M) := (M \times \mathbb{R}^{>0}, t^2 g + dt^2)$, where t denotes the coordinate on the half-line $\mathbb{R}^{>0}$.

Theorem: Suppose $C(M)$ has special holonomy. Then M has the following geometric structures.

Riemannian cones with special holonomy		
Holonomy of $C(M)$	*Geometry of $C(M)$*	*Geometry of M*
$SO(n)$	Riemannian	—
$U(n)$	Kähler	Sasakian
$SU(n)$	Calabi-Yau	Sasaki-Einstein
$Sp(n)$	hyperkähler	3-Sasakian
$Sp(n)Sp(1)$	quaternionic-Kähler	—
G_2	G_2-manifolds	nearly Kähler
$Spin(7)$	$Spin(7)$-manifolds	nearly G_2-manifolds

Killing spinors and parallel spinors

Not essential for understanding of today's talk, because the spinor interpretation will not be used

Recall that we have a "Clifford multiplication map" $TM \otimes \mathfrak{S} \longrightarrow \mathfrak{S}$, where TM is a bundle of tangent vectors on a manifold M, and $\mathfrak{S}$ the bundle of spinors.

DEFINITION: A Killing spinor on M is $\Psi \in \mathfrak{S}$ which satisfies $\nabla_X(\Psi) = \lambda X \Psi$ for all tangent fields $X \in TM$.

DEFINITION: A **parallel spinor** is one which satisfies $\nabla(\Psi) = 0$.

DEFINITION: An Einstein manifold is a Riemannian manifold (M, g) which satisfies $Ric(M) = \lambda g$, where $Ric(M)$ is its Ricci curvature.

Fact 1: Killing spinors on M correspong uniquely to parallel spinors on $C(M)$.

Fact 2: Killing spinors on M exist only if M is an Einstein manifold, with Einstein constant $|\lambda|^2 \geqslant 0$.

REMARK: Similarly, **if M admits a parallel spinor, M is Ricci-Flat** (follows from Weitzenböck formula).

Killing spinors and Riemannian cones

Not essential for understanding of today's talk, because the spinor interpretation will not be used

Remark: In Berger's list, the following holonomies correspond to Ricci-flat manifolds: $SU(n)$, $Sp(n)$, G_2, Spin(7).

Fact 3: $SU(n)$, $Sp(n)$, G_2, Spin(7) admit parallel spinors.

COROLLARY: Sasaki-Einstein, 3-Sasakian, nearly Kähler and nearly G_2-manifolds admit Killing spinors; hence **they are Einstein.**

Proof: Their cones admit a parallel spinor. ∎

Nearly Kähler manifolds

REMARK: The name is confusing, because the nearly Kähler condition, in its strict sense, is **much more restrictive** than the Kähler condition.

The original definition: (Alfred Gray). Let (M, g, I) be a Hermitian almost complex manifold, $\omega \in \Lambda^{1,1}(M)$ its Hermitian form, ∇ the Levi-Civita connection. Then $\nabla\omega$ lies in $\Lambda^1(M) \otimes \Lambda^2(M)$. Gray defined **"nearly Kähler manifolds"** (NK-manifolds) as those that satisfy

$$\nabla\omega \in \Lambda^3(M) \subset \Lambda^1(M) \otimes \Lambda^2(M)$$

($\nabla\omega$ is skew-symmetric).

Trivial remark: In this case $d\omega = \nabla\omega$, because ∇ is torsion-free.

Examples

1. 6-manifolds with parallel G_2 cones.

2. Twistor spaces of positive quaternionic-Kähler manifolds with non-standard complex structure due to Eels and Salamon.

Splitting theorem of P.-A. Nagy

DEFINITION: A **strictly nearly Kähler** manifold is an NK-manifold for which the 3-form $\rho = d\omega$ is **non-degenerate**, that is, the map $TM \xrightarrow{\rho} \Lambda^2 M$, defined as $X \longrightarrow \rho(X,\cdot,\cdot)$, is injective.

REMARK: It is **much more restrictive condition** than the Kähler condition $d\omega = 0$.

THEOREM: **("Splitting theorem", P.-A. Nagy, 2002)**
Let M be a nearly Kähler manifold, in the sense of Gray. Then M is locally a product of the following nearly Kähler types.
1. Homogeneous (symmetric; classified by J.B. Butruille in 2004)
2. Twistor spaces of positive quaternionic-Kähler manifolds
3. 6-dimensional nearly Kähler

REMARK: From this theorem it follows that **strictly nearly Kähler manifolds are products of Einstein ones**.

Some equivalent definitions of NK-manifolds

"A well-known theorem:" (probably due to Friedrich et al)
Let (M, I, ω) be a Hermitian almost complex 6-manifold. Then the following conditions are equivalent.

1. The form $\nabla\omega \in \Lambda^1(M) \otimes \Lambda^2(M)$ is non-zero and **totally skew-symmetric** (that is, $\nabla\omega$ is a 3-form). This means that (M, I, ω) **is nearly Kähler in the sense of Gray, but not Kähler.**

2. The structure group of M admits a reduction to $SU(3)$, that is, there is $(3,0)$-form Ω with $|\Omega| = 1$. **Moreover, one has**

$$d\omega = 3\lambda \operatorname{Re}\Omega, \quad d \operatorname{Im}\Omega = -2\lambda\omega^2$$

where λ is a non-zero real constant.

More equivalent definitions of NK-manifolds (2)

A theorem of H. Baum, T. Friedrich, R. Grunewald, I. Kath :
Let M be a Riemannian 6-manifold. Then the following conditions are equivalent.

1. M **admits a nearly Kähler Hermitian structure.**

2. M **admits a Killing spinor.**

3. The Riemannian cone $C(M)$ **has holonomy** G_2**.**

REMARK: Let M be nearly Kähler. Unless $C(M)$ is flat, and M is S^6, **the almost complex structure is uniquely determined by the metric** (Friedrich). Conversely, **the metric is uniquely determined by the almost complex structure** (M. V.).

6-dimensional NK-manifolds

REMARK: Compact positive quaternionic-Kähler manifolds and their twistors are (conjecturally) symmetric. Hence the only interesting example of "nearly Kähler" is 6-dimensional nearly Kähler manifolds.

In modern literature, "nearly Kähler" usually denotes a 6-dimensional Hermitian manifold with $\nabla\omega$ antisymmetric. We shall always assume "6-dimensional".

A trivial remark: An NK-manifold is never integrable. Indeed, $d\omega^{1,1} = 3\lambda \operatorname{Re}\Omega^{3,0}$. In fact, the Nijenhuis tensor

$$N: \Lambda^{0,1}(M) \longrightarrow \Lambda^{2,0}(M)$$

is **invertible** (unless $\lambda = 0$).

Another trivial remark: If $\lambda = 0$, the NK-equations **degenerate to equations defining Calabi-Yau.**

Examples of nearly Kähler manifolds (all four of them)

1. The sphere S^6. Its cone is $\mathbb{R}^7$.

2 and 3.
$\mathbb{C}P^3$ and the flag variety $F(2,1)$. These are twistor spaces for self-dual Einstein manifolds S^4 and $\mathbb{C}P^2$; we take the Eels-Salamon almost complex structure.

4. $S^3 \times S^3$.

THEOREM: (Butruille)
Any compact homogeneous NK-manifold belongs to this list.

No non-homogeneous compact examples (so far).

Almost complex manifold with totally antisymmetric torsion

THEOREM: (Bismut)
Let (M, I) be a complex manifold, and g a Hermitian metric. Then M **admits a unique connection with totally skew-symmetric torsion** preserving I and g.

THEOREM: (Friedrich-Ivanov)
Let (M, I, ω) be an almost complex Hermitian manifold, and

$$N : \Lambda^2 T^{1,0}(M) \longrightarrow T^{0,1}(M)$$

its Nijenhuis tensor. Consider the 3-linear form $\rho : T^{1,0}(M) \times T^{1,0}(M) \times T^{1,0}(M) \longrightarrow \mathbb{C}$,

$$\rho(x, y, z) := \omega(N(x, y), z)$$

Then M admits a connection ∇ with totally skew-symmetric torsion preserving (ω, I) **if and only if ρ is skew-symmetric.** Moreover, **such a connection is unique.**

Extrema of the volume functional

THEOREM: Let (M, I, ω) be an almost complex Hermitian 6-manifold with nowhere degenerate Nijenhuis tensor, and let $\Psi(I) = \int_M \mathrm{Vol}_I$ be the volume funcional. Then **I is an extremum of Ψ if and only if $d\omega$ lies in $\Lambda^{3,0}(M) \oplus \Lambda^{0,3}(M)$.**

THEOREM: Let (M, I) be a compact almost complex 6-manifold with nowhere degenerate Nijenhuis tensor admitting a Hermitian connection with totally antisymmetric torsion. **I is an extremum of Ψ if and only if (M, I) admits a nearly Kähler metric.**

REMARK: Such a metric is unique (M. V.).