Subtwistor metric on the moduli of hyperkähler manifolds

Misha Verbitsky

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Sub-Riemannian structures

DEFINITION: Let M be a Riemannian manifold and $B \subset TM$ a sub-bundle. A **horizontal path** is a piecewise smooth path $\gamma: [b,a] \longrightarrow M$ tangent to B everywhere. A **sub-Riemannian**, or **Carno-Carathéodory** metric M is

$$d_B(x,y) := \inf_{\gamma \text{ horizontal}} L(\gamma):$$

the infimum of the length $L(\gamma)$ for all horizontal paths connecting x to y.

THEOREM: (Chow-Rashevskii theorem; 1938, 1939) Consider **the Frobenius form** $\Phi: \Lambda^2 B \longrightarrow TM/B$ mapping vector fields $X, Y \in B$ to an image of $[X,Y]$ modulo B. Suppose that Φ is surjective. **Then any two points can be connected by a horizontal path,** and the sub-Riemannian metric d_B is finite.

Properties of sub-Riemannian metrics

Let (M, B, g) be a sub-Riemannian manifold.

CLAIM: Every two points $x, y \in M$ are connected by a smooth, horizontal path γ. Moreover, $d_B(x, y) = \inf_{\gamma \text{ horizontal, smooth}} L(\gamma)$: the sub-Riemannian distance can be taken as infimum of the length for smooth horizontal paths connecting x to y.

THEOREM: (ball-box theorem) An ε-ball in d_B is **asymptotically equivalent to a product of ε-ball in direction of B and ε^2-ball in orthogonal direction.**

COROLLARY: The sub-Riemannian metric **induces the standard topology on M.**

COROLLARY: The Hausdorff dimension of a sub-Riemannian manifold is integer, **and strictly bigger than** $\dim M$**.**

Subtwistor metric

Throughout this talk, H is a real vector space with non-degenerate scalar product of signature $(3, b-3)$, and $\mathrm{Gr}_{++}(H)$ – Grassmannian of 2-dimensional positive oriented planes in H. The space $\mathrm{Gr}_{++}(H)$ is in fact a complex manifold, and it is called **the period space of weight 2 Hodge structures on H.**

DEFINITION: Let $W \subset V$ be a positive 3-dimensional subspace, and $S_W = \mathrm{Gr}_{++}(W) \subset \mathrm{Gr}_{++}(H)$ a 2-dimensional sphere consisting all 2-dimensional oriented planes in W. Then S_w is called **a twistor line**.

CLAIM: Each pair $x, y \in \mathrm{Gr}_{++}(H)$ can be connected by an intersecting chain $S_{W_1}, S_{W_2}, ..., S_{W_n}$ of twistor lines; moreover, $n \leqslant 3$.

DEFINITION: A twistor path on $\mathrm{Gr}_{++}(H)$ is a piecewise smooth path $\gamma : [a,b] \longrightarrow \mathrm{Gr}_{++}(H)$ with each smooth component sitting on a twistor line.

DEFINITION: Fix a Euclidean structure on H, and let g be the corresponding Riemannian metric on $\mathrm{Gr}_{++}(H)$. **Subtwistor metric** $d_{tw}(x,y)$ on $\mathrm{Gr}_{++}(H)$ is defined as $d_{tw}(x,y) := \inf_\gamma L(\gamma)$ where $L(\gamma)$ is a length of the path γ taken with respect to g, and infimum is taken over all subtwistor paths connecting x to y.

Properties of subtwistor metric

QUESTION: Can we connect any pair $x, y \in \mathrm{Gr}_{++}(H)$ with a smooth path tangent to twistor line at each point? Would the infimum of its length give the same metric?

QUESTION: What about the ball-box theorem? What is a shape of a small ε-ball in d_{tw}?

QUESTION: What us the Hausdorff dimension $(\mathrm{Gr}_{++}(H), d_{tw})$?

QUESTION: The definition I gave obviously can be generalized. What is an appropriate generality?

THEOREM: **The subtwistor metric d_{tw} induces the standard topology on $\mathrm{Gr}_{++}(H)$.**

REMARK: Its proof is highly non-trivial; uses a solution of Hilbert's fifth problem on continuous groups.

Hilbert's 5 problem

QUESTION: (Hilbert, 1900)
"How is Lie's concept of continuous groups of transformations of manifolds approachable in our investigation without the assumption of differentiability?"

Answered affirmative by von Neumann, Gleason, Montgomery-Zippin.

THEOREM: Let M be a topological manifold equipped with a continuous group structure. **Then M admits a smooth structure compatible with the group action.**

I will state the Gleason-Palais refinement of this theorem.

Gleason-Palais theorem

DEFINITION: Let M be a topological space. We say that M **has Lebesgue covering dimension** $\leqslant n$ if every open covering of M has a refinement $\{U_i\}$ such that each point of M belongs to at most $n+1$ element of $\{U_i\}$. A **Lebesgue covering dimension** of M (denoted by $\dim M$) is an infimum of all such n.

EXAMPLE: If M is an n-manifold, $\dim M = n$.

CLAIM: If $X \subset M$ is a subset of a topological space, with induced topology, one has $\dim X \leqslant \dim M$.

THEOREM: **(Gleason-Palais)**
Let G be a topological group, which is locally path connected, and has $\dim K < \infty$ for each compact, metrizable subset $K \subset G$. **Then G is homeomorphic to a Lie group.**

Subtwistor norm on a Lie group

REMARK: We define a norm on the group $SO(H)$ compatible with the subtwistor metric on $\mathrm{Gr}_{++}(H)$.

DEFINITION: Let G be a connected component of $SO(H)$ acting on $\mathrm{Gr}_{++}(H)$ in a susual way. We define **subtwistor norm** on G in such a way that **the bijective map** $(G/G_0, \|\cdot\|_{tw}) \longrightarrow (\mathrm{Gr}_{++}(H), d_{tw})$ **is continuous**, where $G_0 \subset G$ is a stabilizer of a point $V \in \mathrm{Gr}_{++}(H)$.

DEFINITION: An **elementary transform** is an element $h \in G$ fixing a codimension 2 subspace $V_1 \subset V$ of signature $(1, n-3)$. **An elementary decomposition** of $h \in G$ is a decomposition $h = h_1 h_2 ... h_n$, where h_i are elementary transforms. Define the **subtwistor norm** on G as $\|h\|_{tw} := \inf(\|h_1\| + \|h_2\| + ... + \|h_n\|)$, where the infinum is taken over all elementary decompositions $h = h_1 h_2 ... h_n$.

CLAIM: The action of $(G, \|\cdot\|_{tw})$ on $(\mathrm{Gr}_{++}(H), d_{tw})$ is continuous, and induces a homeomorphism $(G/G_0, \|\cdot\|_{tw}) \longrightarrow (\mathrm{Gr}_{++}(H), d_{tw})$.

Transformation groups and subtwistor metrics

THEOREM: The subtwistor metric d_{tw} induces the standard topology on $\mathrm{Gr}_{++}(H)$.

Step 1: Since $\mathrm{Gr}_{++}(H) \cong (G/G_0, \|\cdot\|_{tw})$, it suffices to show that the subtwistor norm defines the usual topology on G.

Step 2: Let $\|\cdot\|$ be the usual norm on G. Since $\|\cdot\|_{tw} \geqslant \|\cdot\|$, **the identity map $(G, \|\cdot\|_{tw}) \longrightarrow (G, \|\cdot\|)$ is continuous.**

Step 3: (Brouwer's invariance of domain theorem):
Let $X \xrightarrow{f} Y$ be a continuous, bijective map of Hausdorff manifolds. Then f is a homeomorphism. Apply this to the identity map $(G, \|\cdot\|_{tw}) \longrightarrow G$. To prove that it is a homeomorphism, **it remains to show that $(G, \|\cdot\|_{tw})$ is a manifold.**

Step 4: Since a bijective continuous map from a compact is a homeomorphism, the identity map $(G, \|\cdot\|_{tw}) \longrightarrow (G, \|\cdot\|)$ is a homeomorphism on compacts. Therefore, the Lebesgue covering dimension of any compact is the same in $(G, \|\cdot\|_{tw})$ and in $(G, \|\cdot\|)$, hence finite. Path connectedness of $(G, \|\cdot\|_{tw})$ is clear from its construction. Then **Gleason-Palais implies that $G, \|\cdot\|_{tw}$ is a manifold.** ■

Teichmüller space

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

DEFINITION: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

REMARK: This terminology is **standard for curves.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

DEFINITION: A complex surface is called **a K3 surface** if it a deformation of a Kummer surface. K3 surface is also hyperkähler.

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic 2-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

REMARK: Usually, one says "hyperkähler manifold" meaning "a compact, Kähler, holomorphically symplectic manifold".

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.**

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is simple and hyperkähler. Then $C \int_M \eta^{2n} = q(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$ and $C > 0$.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by this relation uniquely, up to a sign.**

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, \ \ q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = \mathrm{Gr}_{++}(H^2(M,\mathbb{R}), q)$

THEOREM: (Bogomolov) Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. **Then the period map $P:$ Teich $\longrightarrow \mathbb{P}er$ is locally a diffeomorphism.**

Global Torelli theorem

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: Let M be a hyperkähler manifold, Teich its Teichmüller space, and Teich_b the quotient of Teich by $\sim$. **Then the period map $P: \text{Teich}_b \longrightarrow \mathbb{P}er$ induces a diffeomorphism on each connected component.**

REMARK: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l,l) = 0, \quad q(l, \bar{l}) > 0.\}$$

is identified with $Gr_{++}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

Proof of Global Torelli theorem

DEFINITION: Let (M, I, J, K) be a hyperkähler manifold. **A hyperkähler 3-plane** in $H^2(M,\mathbb{R})$ is a positive oriented 3-dimensional subspace W, generated by $\omega_I, \omega_J, \omega_K$.

REMARK: The set of oriented 2-dimensional planes in W is identified with $S^2 = \mathbb{C}P^1$. It is called **the twistor family** of a hyperkähler structure. A point in the twistor family corresponds to a complex structure $aI + bJ + cK \in \mathbb{H}$, with $a^2 + b^2 + c^2 = 1$. We call the corresponding $\mathbb{C}P^1 \subset \mathrm{Teich}$ **the twistor lines**.

DEFINITION: We call a subspace $R \subset H^2(M,\mathbb{R})$ **irrational** if $R^\perp \cap H^2(M,\mathbb{Q})$ is empty.

THEOREM: Let $S \subset \mathbb{P}\mathrm{er}$ be a twistor line corresponding to an irrational plane $\mathrm{Gr}_{+++}(H^2(M,\mathbb{R}))$. **Then it can be lifted to** Teich **with each of the irrational point in its preimage.**

COROLLARY: The period map $\mathrm{Teich}_b \longrightarrow \mathbb{P}\mathrm{er}$ **is an isometry with respect to the subtwistor metrics.**

REMARK: Now the global Torelli follows, because (being an isometry) it is also a covering.

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

is identified with $SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M,\mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M,\mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l,\bar{l})$ implies that $l \cap H^2(M,\mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l,\bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M,\mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■