Subtwistor metric
on the moduli of hyperkähler manifolds

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Sub-Riemannian structures

**DEFINITION:** Let $M$ be a Riemannian manifold and $B \subset TM$ a sub-bundle. A horizontal path is a piecewise smooth path $\gamma : [b, a] \rightarrow M$ tangent to $B$ everywhere. A sub-Riemannian, or Carno-Carathéodory metric $M$ is

$$d_B(x, y) := \inf_{\gamma \text{ horizontal}} L(\gamma) :$$

the infimum of the length $L(\gamma)$ for all horizontal paths connecting $x$ to $y$.

**THEOREM:** (Chow-Rashevskii theorem; 1938, 1939)
Consider the Frobenius form $\Phi : \Lambda^2 B \rightarrow TM/B$ mapping vector fields $X, Y \in B$ to an image of $[X, Y]$ modulo $B$. Suppose that $\Phi$ is surjective. Then any two points can be connected by a horizontal path, and the sub-Riemannian metric $d_B$ is finite.
Properties of sub-Riemannian metrics

Let \((M,B,g)\) be a sub-Riemannian manifold.

**CLAIM:** Every two points \(x,y \in M\) are connected by a smooth, horizontal path \(\gamma\). Moreover, \(d_B(x,y) = \inf_{\gamma \text{ horizontal, smooth}} L(\gamma)\): the sub-Riemannian distance can be taken as infimum of the length for smooth horizontal paths connecting \(x\) to \(y\).

**THEOREM:** (ball-box theorem) An \(\varepsilon\)-ball in \(d_B\) is asymptotically equivalent to a product of \(\varepsilon\)-ball in direction of \(B\) and \(\varepsilon^2\)-ball in orthogonal direction.

**COROLLARY:** The sub-Riemannian metric induces the standard topology on \(M\).

**COROLLARY:** The Hausdorff dimension of a sub-Riemannian manifold is integer, and strictly bigger than \(\dim M\).
Subtwistor metric

Throughout this talk, $H$ is a real vector space with non-degenerate scalar product of signature $(3, b-3)$, and $\text{Gr}_{++}(H)$ – Grassmannian of 2-dimensional positive oriented planes in $H$. The space $\text{Gr}_{++}(H)$ is in fact a complex manifold, and it is called **the period space of weight 2 Hodge structures on** $H$.

**DEFINITION:** Let $W \subset V$ be a positive 3-dimensional subspace, and $S_W = \text{Gr}_{++}(W) \subset \text{Gr}_{++}(H)$ a 2-dimensional sphere consisting all 2-dimensional oriented planes in $W$. Then $S_W$ is called a **twistor line**.

**CLAIM:** Each pair $x, y \in \text{Gr}_{++}(H)$ can be connected by an intersecting chain $S_W_1, S_W_2, ..., S_W_n$ of twistor lines; moreover, $n \leq 3$.

**DEFINITION:** A **twistor path** on $\text{Gr}_{++}(H)$ is a piecewise smooth path $\gamma : [a, b] \rightarrow \text{Gr}_{++}(H)$ with each smooth component sitting on a twistor line.

**DEFINITION:** Fix a Euclidean structure on $H$, and let $g$ be the corresponding Riemannian metric on $\text{Gr}_{++}(H)$. **Subtwistor metric** $d_{tw}(x, y)$ on $\text{Gr}_{++}(H)$ is defined as $d_{tw}(x, y) := \inf_{\gamma} L(\gamma)$ where $L(\gamma)$ is a length of the path $\gamma$ taken with respect to $g$, and infimum is taken over all subtwistor paths connecting $x$ to $y$. 


Properties of subtwistor metric

**QUESTION:** Can we connect any pair $x, y \in \text{Gr}_{++}(H)$ with a smooth path tangent to twistor line at each point? Would the infimum of its length give the same metric?

**QUESTION:** What about the ball-box theorem? What is a shape of a small $\varepsilon$-ball in $d_{tw}$?

**QUESTION:** What is the Hausdorff dimension $(\text{Gr}_{++}(H), d_{tw})$?

**QUESTION:** The definition I gave obviously can be generalized. What is an appropriate generality?

**THEOREM:** The subtwistor metric $d_{tw}$ induces the standard topology on $\text{Gr}_{++}(H)$.

**REMARK:** Its proof is highly non-trivial; uses a solution of Hilbert’s fifth problem on continuous groups.
Hilbert’s 5 problem

**QUESTION: (Hilbert, 1900)**
“How is Lie’s concept of continuous groups of transformations of manifolds approachable in our investigation without the assumption of differentiability?”

**Answered affirmative** by von Neumann, Gleason, Montgomery-Zippin.

**THEOREM:** Let $M$ be a topological manifold equipped with a continuous group structure. Then $M$ admits a smooth structure compatible with the group action.

I will state the Gleason-Palais refinement of this theorem.
Gleason-Palais theorem

**DEFINITION:** Let $M$ be a topological space. We say that $M$ has **Lebesgue covering dimension** $\leq n$ if every open covering of $M$ has a refinement $\{U_i\}$ such that each point of $M$ belongs to at most $n + 1$ element of $\{U_i\}$. A **Lebesgue covering dimension** of $M$ (denoted by $\dim M$) is an infimum of all such $n$.

**EXAMPLE:** If $M$ is an $n$-manifold, $\dim M = n$.

**CLAIM:** If $X \subset M$ is a subset of a topological space, with induced topology, one has $\dim X \leq \dim M$.

**THEOREM:** *(Gleason-Palais)*
Let $G$ be a topological group, which is locally path connected, and has $\dim K < \infty$ for each compact, metrizable subset $K \subset G$. Then $G$ is homeomorphic to a Lie group.
**Subtwistor norm on a Lie group**

**REMARK:** We define a norm on the group $SO(H)$ compatible with the subtwistor metric on $\Gr_{++}(H)$.

**DEFINITION:** Let $G$ be a connected component of $SO(H)$ acting on $\Gr_{++}(H)$ in a usual way. We define **subtwistor norm** on $G$ in such a way that the bijective map $(G/G_0, \| \cdot \|_{tw}) \to (\Gr_{++}(H), d_{tw})$ is **continuous**, where $G_0 \subset G$ is a stabilizer of a point $V \in \Gr_{++}(H)$.

**DEFINITION:** An **elementary transform** is an element $h \in G$ fixing a codimension 2 subspace $V_1 \subset V$ of signature $(1, n - 3)$. **An elementary decomposition** of $h \in G$ is a decomposition $h = h_1 h_2 \ldots h_n$, where $h_i$ are elementary transforms. Define the **subtwistor norm** on $G$ as $\|h\|_{tw} := \inf(\|h_1\| + \|h_2\| + \ldots + \|h_n\|)$, where the infimum is taken over all elementary decompositions $h = h_1 h_2 \ldots h_n$.

**CLAIM:** The action of $(G, \| \cdot \|_{tw})$ on $(\Gr_{++}(H), d_{tw})$ is **continuous**, and induces a homeomorphism $(G/G_0, \| \cdot \|_{tw}) \to (\Gr_{++}(H), d_{tw})$. 

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**Transformation groups and subtwistor metrics**

**THEOREM:** The subtwistor metric $d_{tw}$ induces the standard topology on $\text{Gr}_{++}(H)$.

**Step 1:** Since $\text{Gr}_{++}(H) \cong (G/G_0, \| \cdot \|_{tw})$, it suffices to show that the subtwistor norm defines the usual topology on $G$.

**Step 2:** Let $\| \cdot \|$ be the usual norm on $G$. Since $\| \cdot \|_{tw} \geq \| \cdot \|$, the identity map $(G, \| \cdot \|_{tw}) \rightarrow (G, \| \cdot \|)$ is continuous.

**Step 3:** (Brouwer’s invariance of domain theorem):
Let $X \xrightarrow{f} Y$ be a continuous, bijective map of Hausdorff manifolds. Then $f$ is a homeomorphism. Apply this to the identity map $(G, \| \cdot \|_{tw}) \rightarrow G$. To prove that it is a homeomorphism, it remains to show that $(G, \| \cdot \|_{tw})$ is a manifold.

**Step 4:** Since a bijective continuous map from a compact is a homeomorphism, the identity map $(G, \| \cdot \|_{tw}) \rightarrow (G, \| \cdot \|)$ is a homeomorphism on compacts. Therefore, the Lebesgue covering dimension of any compact is the same in $(G, \| \cdot \|_{tw})$ and in $(G, \| \cdot \|)$, hence finite. Path connectedness of $(G, \| \cdot \|_{tw})$ is clear from its construction. Then Gleason-Palais implies that $G, \| \cdot \|_{tw}$ is a manifold. ■
**Teichmüller space**

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\text{Comp}$ the space of complex structures on $M$, and let $\text{Teich} := \text{Comp} / \text{Diff}_0(M)$. We call it the **Teichmüller space**.

**Remark:** $\text{Teich}$ is a finite-dimensional complex space (Kodaira-Spencer-Kuranishi-Douady), but often non-Hausdorff.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M) / \text{Diff}_0(M)$ the **mapping class group**. The **moduli space of complex structures on** $M$ is a connected component of $\text{Teich}/\Gamma$.

**Remark:** This terminology is standard for curves.
Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold is holomorphically symplectic: $\omega J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**EXAMPLE:** Take a 2-dimensional complex torus $T$, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called a Kummer surface. It is holomorphically symplectic.

**DEFINITION:** A complex surface is called a K3 surface if it a deformation of a Kummer surface. K3 surface is also hyperkähler.
Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic 2-form.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**REMARK:** Usually, one says “hyperkähler manifold” meaning “a compact, Kähler, holomorphically symplectic manifold”.

**DEFINITION:** A hyperkähler manifold $M$ is called simple if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and dim $M = 2n$, where $M$ is simple and hyperkähler. Then $C \int_M \eta^{2n} = q(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$ and $C > 0$.

**Definition:** This form is called Bogomolov-Beauville-Fujiki form. It is defined by this relation uniquely, up to a sign.
The period map

Remark: For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called the period map.

Remark: $P$ maps $\text{Teich}$ into an open subset of a quadric, defined by $\text{Per} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, \quad q(l, \bar{l}) > 0 \}$. It is called the period space of $M$.

Remark: $\text{Per} = \text{Gr}_{++}(H^2(M, \mathbb{R}), q)$

Theorem: (Bogomolov) Let $M$ be a simple hyperkähler manifold, and $\text{Teich}$ its Teichmüller space. Then the period map $P : \text{Teich} \longrightarrow \text{Per}$ is locally a diffeomorphism.
Global Torelli theorem

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are non-separable (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** Let $M$ be a hyperkähler manifold, Teich its Teichmüller space, and Teich$_b$ the quotient of Teich by $\sim$. Then the period map $P : \text{Teich}_b \to \text{Per}$ induces a diffeomorphism on each connected component.

**REMARK:** The period space

$$\text{Per} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, \quad q(l, \bar{l}) > 0 \}$$

is identified with $Gr_{++}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$. 
Proof of Global Torelli theorem

**DEFINITION:** Let \((M, I, J, K)\) be a hyperkähler manifold. A hyperkähler 3-plane in \(H^2(M, \mathbb{R})\) is a positive oriented 3-dimensional subspace \(W\), generated by \(\omega_I, \omega_J, \omega_K\).

**REMARK:** The set of oriented 2-dimensional planes in \(W\) is identified with \(S^2 = \mathbb{C}P^1\). It is called the twistor family of a hyperkähler structure. A point in the twistor family corresponds to a complex structure \(aI + bJ + cK \in \mathbb{H}\), with \(a^2 + b^2 + c^2 = 1\). We call the corresponding \(\mathbb{C}P^1 \subset \text{Teich}\) the twistor lines.

**DEFINITION:** We call a subspace \(R \subset H^2(M, \mathbb{R})\) irrational if \(R^\perp \cap H^2(M, \mathbb{Q})\) is empty.

**THEOREM:** Let \(S \subset \text{Per}\) be a twistor line corresponding to an irrational plane \(\text{Gr}_{+++}(H^2(M, \mathbb{R}))\). Then it can be lifted to \(\text{Teich}\) with each of the irrational point in its preimage.

**COROLLARY:** The period map \(\text{Teich}_b \rightarrow \text{Per}\) is an isometry with respect to the subtwistor metrics.

**REMARK:** Now the global Torelli follows, because (being an isometry) it is also a covering.
Period space as a Grassmannian of positive 2-planes

**PROPOSITION:** The period space

\[ \mathbb{P}_{\text{Per}} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \}. \]

is identified with \( SO(b_2-3, 3)/SO(2) \times SO(b_2-3, 1) \), which is a Grassmannian of positive oriented 2-planes in \( H^2(M, \mathbb{R}) \).

**Proof.** **Step 1:** Given \( l \in \mathbb{P}H^2(M, \mathbb{C}) \), the space generated by \( \text{Im} \ l, \text{Re} \ l \) is 2-dimensional, because \( q(l, l) = 0, q(l, \bar{l}) \) implies that \( l \cap H^2(M, \mathbb{R}) = 0 \).

**Step 2:** This 2-dimensional plane is positive, because \( q(\text{Re} l, \text{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0 \).

**Step 3:** Conversely, for any 2-dimensional positive plane \( V \in H^2(M, \mathbb{R}) \), the quadric \( \{ l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0 \} \) consists of two lines; a choice of a line is determined by orientation. ■