

Generalization of Inoue surfaces by Oeljeklaus-Toma and number theory

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Non-Kählerian aspects of complex geometry

February 21, 2011

Plan.

- 1. Motivation:** classification of 2-dimensional solvmanifolds, Inoue surfaces, Bogomolov's theorem.
- 2. Number theory:** global and local fields, absolute value function, complex and real embeddings, the Dirichlet's unit theorem.
- 3. Inoue surfaces of class S^0 .** Curves on Inoue surfaces.
- 4. Oeljeklaus-Toma manifolds.** Subvarieties in Oeljeklaus-Toma manifolds.
- 5. The adèle ring and the strong approximation theorem.**

Solvmanifolds

DEFINITION: Let M be a smooth manifold equipped with a transitive action of solvable Lie group. Then M is called **a solvmanifold**.

REMARK: All solvmanifolds are obtained as quotient spaces, $M = G/H$.

DEFINITION: An **integrable complex structure** on a real Lie algebra \mathfrak{g} is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: Right-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: A **complex solvmanifold** is a solvmanifold $M = G/H$ equipped with a complex structure, in such a way that G has a right-invariant complex structure, and the projection $G \rightarrow M$ is holomorphic.

REMARK: **Solvmanifolds are usually non-homogeneous** (as complex manifolds).

Examples of 2-dimensional solvmanifolds

REMARK: “A surface” here would always mean “a compact complex manifold of complex dimension 2”.

DEFINITION: Let T, T' be elliptic curves. **Kodaira surface** is a locally trivial holomorphic fibration over T with fiber T' and non-trivial Chern class.

A remark on terminology: These are “primary” Kodaira surfaces. “Secondary” ones are obtained by taking finite unramified quotients.

REMARK: The Kodaira surface is diffeomorphic to a quotient $S^1 \times (G/G_{\mathbb{Z}})$ where G is a 3-dimensional Heisenberg group. In particular, **Kodaira surface is a complex nilmanifold.**

Inoue surfaces

DEFINITION: ("Bogomolov's theorem") **Inoue surface** is a complex surface without curves and with $b_2 = 0$.

REMARK: Original definition of Inoue was constructive, in terms of explicit action by matrices, and the above result is a theorem proven by Bogomolov in 1976.

Bogomolov, F. A. *Classification of surfaces of class VII₀ with $b_2 = 0$* . Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 2, 273-288, 469.

Math. Reviews: *...This unreadable paper contains several new ideas and claims two important results; unfortunately many of the arguments in the proof are subject to doubt. It would seem highly desirable to know whether proofs along the lines given here can be made to work. ... Without going into detailed criticism of the author's written style, the reviewer would like to comment that in the parts of the paper which he has been able to understand the author's disorganised shorthand and bestial notation put a burden on the reader which he considers unacceptable in a published paper. – Miles Reid*

Bogomolov's proof uses the action of the Galois group $[\mathbb{C} : \mathbb{Q}]$.

History of Inoue surfaces

In 1991, a new proof appeared, based on Yang-Mills theory:

Li, J.; Yau, S.-T.; Zheng, F. *A simple proof of Bogomolov's theorem on class VII_0 surfaces with $b_2 = 0$* . Illinois J. Math. 34 (1990), no. 2, 217-220.

This proof was wrong.

Finally, correct proofs were obtained.

Teleman, Andrei Dumitru *Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces*. Internat. J. Math. 5 (1994), no. 2, 253-264.

Li, Jun; Yau, Shing-Tung; Zheng, Fangyang *On projectively flat Hermitian manifolds*. Comm. Anal. Geom. 2 (1994), no. 1, 103-109.

Complex solvmanifolds of dimension 2

THEOREM: (Hasegawa) Let M be a complex surface which is diffeomorphic to a solvmanifold. Then M is (up to a finite unramified quotient) **isomorphic to one the following.**

1. Compact complex torus
2. Kodaira surface
3. Inoue surface.

This theorem directly follows from Bogomolov's theorem, Hasegawa's result on Kähler solvmanifolds, and Kodaira's classification.

To define the Inoue surfaces explicitly, we use **the number theory.**

Normed fields

DEFINITION: An **absolute value** on a field k is a function $|\cdot| : k \longrightarrow \mathbb{R}^{\geq 0}$, satisfying the following

1. **Zero:** $|x| = 0 \Leftrightarrow x = 0$.
2. **Multiplicativity:** $|xy| = |x||y|$.
3. **There exists $c > 0$ such that $|\cdot|^c$ satisfies the triangle inequality.**

EXAMPLE: The usual absolute value on \mathbb{Q} , \mathbb{R} , \mathbb{C} .

EXAMPLE: Let p be a prime number, and $m, n \in \mathbb{Z}$ coprime with p . Define **p -adic absolute value** on \mathbb{Q} via $|\frac{m}{n}p^k| := p^{-k}$.

REMARK: p -adic absolute value satisfies an additional “non-archimedean axiom”: $|x+y| \leq \max(|x|, |y|)$. Such absolute values are called **non-archimedean**.

REMARK: Any power of non-archimedean absolute value is again non-archimedean, and satisfies the triangle inequality.

Normed fields and topology

DEFINITION: Let $|\cdot|$ be an absolute value on a field F . Consider topology on F with open sets generated by

$$B_\varepsilon(x) := \{y \in k \mid |x - y| < \varepsilon\}.$$

Absolute values are called **equivalent** if they induce the same topology.

THEOREM: Absolute values $|\cdot|_1, |\cdot|_2$ are equivalent if and only if $|\cdot|_1 = |\cdot|_2^c$ for some $c > 0$.

THEOREM: (Ostrowski) Every absolute value on \mathbb{Q} is equivalent to the usual ("archimedean") one or to p -adic one.

DEFINITION: A **completion** of a field k under an absolute value $|\cdot|$ is a completion of k in a metric $|\cdot|^c$, where $c > 0$ is a constant such that $|\cdot|^c$ satisfies the triangle inequality.

REMARK: A completion of a field is again a field.

EXAMPLE: A completion of \mathbb{Q} under the p -adic absolute value is called a **field of p -adic numbers**, denoted \mathbb{Q}_p .

Local fields

DEFINITION: A **finite extension** $K:k$ of fields is a field $K \supset k$ which is finite-dimensional as a vector space over k . A **number field** is a finite extension of \mathbb{Q} . A **functional field** is a finite extension of $\mathbb{F}_p(t)$. A **global field** is a number or functional field. A **local field** is a completion of a global field under a non-trivial absolute value.

THEOREM: Let \bar{k} be a field which is complete and locally compact under some absolute value. **Then \bar{k} is a local field.**

DEFINITION: Let $K:k$ be a finite extension, and $x \in K$. Consider the multiplication by x as a k -linear endomorphism of K . Define **the norm** $N_{K/k}(x)$ as a determinant of this operator.

REMARK: Norm defines a homomorphism of multiplicative groups $K^* \longrightarrow k^*$.

REMARK: For Galois extensions, the norm $N_{K/k}(x)$ **is a product of all elements conjugate to x .**

THEOREM: Let $\bar{K}:\bar{k}$ be a finite extension of local fields, degree n . **Then an absolute value on \bar{k} is uniquely extended to \bar{K} .** Moreover, **this extension is expressed as $|x| := |N_{\bar{K}/\bar{k}}(x)|^{\frac{1}{n}}$.**

Absolute values and extensions of global fields

CLAIM: Let A, B be extensions of a field k , $\text{char } k = 0$, where $A:k$ is finite. Consider $A \otimes_k B$ as a k -algebra. **Then $A \otimes_k B$ is a direct sum of fields, containing A and B .**

THEOREM: Let k be a number field, $|\cdot|$ an absolute value, $K:k$ a finite extension, and \bar{k} – its completion. Consider a decomposition $K \otimes_k \bar{k}$ into a direct sum of fields $K \otimes_k \bar{k} := \bigoplus_i \bar{K}_i$. **Then each extension of an absolute value $|\cdot|$ from k to K is induced from some \bar{K}_i , and all such extensions are non-equivalent.**

REMARK: When $k = \mathbb{Q}$, and $|\cdot|$ is the usual (archimedean) absolute value, we obtain that all K_i are extensions of \mathbb{R} , that is, isomorphic to \mathbb{R} or \mathbb{C} . This gives

COROLLARY: **For each number field K of degree n over \mathbb{Q} , there exists only a finite number of different homomorphisms $K \hookrightarrow \mathbb{C}$, all of them injective. Denote by s the number of embeddings whose image lies in $\mathbb{R} \subset \mathbb{C}$ (such an embedding is called **real**), and $2t$ the number of embeddings whose image does not lie in \mathbb{R} (“**complex embeddings**”). Then $s + 2t = n$.**

Dirichlet unit theorem

DEFINITION: Let $K:\mathbb{Q}$ be a number field of degree n . **The ring of integers** $\mathcal{O}_K \subset K$ is an integral closure of \mathbb{Z} in K , that is, the set of all roots in K of monic polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots + a_0$ with integer coefficients $a_i \in \mathbb{Z}$.

CLAIM: An additive group \mathcal{O}_K^+ is a finitely generated abelian group of rank n .

DEFINITION: A **unit** of a ring \mathcal{O}_K is an element $u \in \mathcal{O}_K$, such that u^{-1} also belongs to \mathcal{O}_K .

REMARK: $u \in \mathcal{O}_K$ is a unit if and only if the norm $N_{K/\mathbb{Q}}(x) \in \mathbb{Z}$ is invertible, that is, $N_{K/\mathbb{Q}}(x) = \pm 1$.

Dirichlet's unit theorem: Let K be a number field which has s real embeddings and $2t$ complex ones. Then **the group of units \mathcal{O}_K^* is isomorphic to $G \times \mathbb{Z}^{t+s-1}$** , where G is a finite group of roots of unity contained in K . Moreover, if $s > 0$, one has $G = \pm 1$.

REMARK: For a quadratic field, the group of units is a group of solutions of Pell's equation.

Cubic fields and complex surfaces

Let $K:\mathbb{Q}$ be a cubic extension of \mathbb{Q} which has 2 complex embeddings $\tau, \bar{\tau}$ and one real one σ (such an extension is obtained by adding all roots of a cubic polynomial which has exactly one real root).

REMARK: Due to Dirichlet theorem, \mathcal{O}_K^* is isomorphic to $\mathbb{Z} \times \{\pm 1\}$. Let $\mathcal{O}_K^{*,+} := \sigma^{-1}(\mathbb{R}^{>0}) \cap \mathcal{O}_K^*$. Then **the group $\mathcal{O}_K^{*,+}$ is isomorphic to \mathbb{Z} .**

Consider the action of $\mathcal{O}_K^+ \cong \mathbb{Z}^3$ on $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$

$$\rho^+(x)(z, t) := (z + \tau(x), t + \sigma(x)).$$

Let Γ be a semidirect product $\mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,+}$, defined from the natural action of $\mathcal{O}_K^{*,+}$ on \mathcal{O}_K^+ . **Define an action of Γ on $\mathbb{C} \times H$, where H is an upper halfplane**, as follows.

The subgroup $\mathcal{O}_K^+ \subset \Gamma$ acts on $\mathbb{C} \times H = \mathbb{C} \times \mathbb{R} \times \mathbb{R}^{>0}$ by translations as above (trivially on the last argument), and $\mathcal{O}_K^{*,+}$ acts multiplicatively as

$$\rho^*(\xi)(z, z') := (\tau(\xi)z, \sigma(\xi)z').$$

Inoue surfaces of type S^0

DEFINITION: The **Inoue surface of type S^0** is a quotient $(\mathbb{C} \times H)/\Gamma$.

Its properties: 1. It is a compact, complex solvmanifold

2. Inoue surface **admits a flat connection preserving the complex structure** (by construction).

3. Its cohomology are the same as of $S^3 \times S^1$.

THEOREM: The Inoue surface $M := (\mathbb{C} \times H)/\Gamma$ **has no complex curves**

Proof. Step 1: Consider on $\mathbb{C} \times H$ a function $\varphi(z, z') := \log \operatorname{Im}(z')$. Since Γ multiplies $\operatorname{Im}(z')$ by a number, **the form $d\varphi$ is Γ -invariant**. Let θ be the corresponding 1-form on M .

Step 2: The 2-form $\omega_0 := d(I\theta)$ has Hodge type (1,1) and **is positive definite on the leaves of the foliation $\{z\} \times H \subset \mathbb{C} \times H$** . Indeed,

$$\omega_0 = \sqrt{-1} \partial \bar{\partial} \log \varphi = \sqrt{-1} \frac{dz' \wedge d\bar{z}'}{|\operatorname{Im} z'|^2},$$

where ω_0 is the Poincare metric on H .

Curves on Inoue surfaces

Step 3: Let $\Sigma \subset TM$ be the null-space of the form ω_0 . **It is a holomorphic, involutive foliation**, whose leaves are obtained from $\mathbb{C} \times \{z'\} \subset \mathbb{C} \times H$

Step 4: For any complex curve C on M , $\int_C \omega_0 = 0$, because ω_0 is exact. Therefore, C is tangent to a leaf of Σ . **It remains to show that Σ has no compact leaves.**

Step 5: Let Σ_0 be a leaf of Σ . Its preimage in $\mathbb{C} \times H$ contains the set

$$\tilde{\Sigma}_0 := \bigcup_{z \in \mathbb{C}, \zeta \in \mathcal{O}_K^+} \left(z, (z' + \sigma(\zeta)) \right)$$

where $z' \in H$ is a fixed point. Since the image of σ is dense in \mathbb{R} , **the closure $\tilde{\Sigma}_0$ contains $\mathbb{C} \times \mathbb{R} \times \text{Im}(z')$.**

Step 6: Therefore, **the closure $\Sigma_0 \subset M$ is at least 3-dimensional**, hence **Σ has no compact leaves.**

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embeddings denoted $\tau_i, \bar{\tau}_i$ and s real ones denoted σ_i , $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \prod_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+}$ a free abelian subgroup $\mathcal{O}_K^{*,U}$ of rank s such that the quotient $\mathbb{R}^s / \mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to \mathbb{R}^t as $\xi \rightarrow (\log(\sigma_1(\xi)), \dots, \log(\sigma_t(\xi)))$. Let $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: An **Oeljeklaus-Toma manifold** is a quotient $\mathbb{C}^t \times H^s / \Gamma$, where \mathcal{O}_K^+ acts on $\mathbb{C}^t \times H^s$ as

$$\zeta(x_1, \dots, x_t, y_1, \dots, y_s) = \left(x_1 + \tau_1(\zeta), \dots, x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), \dots, y_s + \sigma_s(\zeta) \right),$$

and $\mathcal{O}_K^{*,U}$ as

$$\xi(x_1, \dots, x_t, y_1, \dots, y_s) = \left(x_1, \dots, x_t, \sigma_1(\xi)y_1, \dots, \sigma_t(\xi)y_t \right)$$

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times H^s / \Gamma$ is a **compact complex manifold**, without any non-constant meromorphic functions. When $t = 1$, it is locally conformally Kähler. When $s = 1, t = 1$, it is an Inoue surface of class S^0 .

Complex geometry of Oeljeklaus-Toma manifolds

THEOREM: (Ornea-V.) Let K be a number field which has s real embeddings and $2t$ complex ones, $t = 1$, $s > 0$. **Then the corresponding Oeljeklaus-Toma manifold has no non-trivial complex subvarieties.**

Proof. Step 1: Consider on $\mathbb{C} \times H^t$ a function $\varphi(z, z_1, \dots, z_s) := \prod_i \text{Im}(z_i)$. Since Γ multiplies $\text{Im}(z_i)$ by a number, **the form $d \log \varphi$ is Γ -invariant.** Let θ denote the corresponding 1-form on $M = \mathbb{C} \times H^s / \Gamma$.

Step 2: The 2-form $\omega_0 := d(I\theta)$ has Hodge type (1,1) and **positive definite on the leaves of the foliation $\{z\} \times H^t \subset \mathbb{C} \times H^t$**

$$\omega_0 = \sqrt{-1} \partial \bar{\partial} \log \varphi = \sqrt{-1} \sum_i \frac{dz_i \wedge d\bar{z}_i}{|\text{im } z_i|^2}.$$

Also, $\omega_0 \geq 0$.

Step 3: Let $\Sigma \subset TM$ be the null-foliation of ω_0 (the foliation generated by the null eigenspace). **It is a holomorphic, involutive, smooth 1-dimensional foliation,** with the leaves which are obtained from $\mathbb{C} \times \{(z_1, \dots, z_s)\} \subset \mathbb{C} \times H^s$.

Step 4: For any complex k -dimensional subvariety $C \subset M$, the integral $\int_C \omega_0^k = 0$, because ω_0 is exact. Therefore, C is at each point tangent to a leaf of Σ . **Since Σ is 1-dimensional, this means that C contains at least one leaf of Σ .**

Step 5: It remains to show that any variety which contains a leaf of Σ coincides with M .

Step 6: Let Σ_0 be a leaf of Σ . Its preimage in $\mathbb{C} \times H^s$ contains a set

$$\tilde{\Sigma}_0(z_1, \dots, z_s) := \bigcup_{z \in \mathbb{C}, \zeta \in \mathcal{O}_K^+} \left(z, (z_1 + \sigma_1(\zeta), \dots, z_s + \sigma_s(\zeta)) \right)$$

where $z_1, \dots, z_s \in H^s$ is some fixed point.

Step 7: We reduced the theorem to the following statement

CLAIM: A closure of $\tilde{\Sigma}_0(z_1, \dots, z_s)$ contains a set

$$Z_{\alpha_1, \dots, \alpha_s} := \{(\zeta, \zeta_1, \dots, \zeta_s) \mid \text{im } \zeta_i = \alpha_i, i = 1, \dots, s\}$$

where $\alpha_i = \text{im } z_i$.

Indeed, **the smallest complex subspace containing $T_x Z_{\alpha_1, \dots, \alpha_s}$ is $T_x M$.**

The adèle ring

The previous claim is immediately implied by the following statement, applied to **the set ρ_1, \dots, ρ_m of all real embeddings**.

Theorem 1 Let $K:\mathbb{Q}$ be a number field with has $2t$ complex embeddings $\tau_1, \bar{\tau}_1, \dots$ and s real ones, $\sigma_1, \dots, \sigma_t, \rho_1, \dots, \rho_m$ – embeddings K to \mathbb{C} or \mathbb{R} , and each of τ_i and σ_i appears once, except one. Consider the map $R : K \longrightarrow \mathbb{R}^a \times \mathbb{C}^b, R(\xi) := \rho_1(\xi), \dots, \rho_m(\xi)$. **Then the image of \mathcal{O}_K is dense in $\mathbb{R}^a \times \mathbb{C}^b$.**

The proof is based on the **strong approximation theorem** (which is a “modern version” of Chinese remainders theorem).

DEFINITION: Adelic group \mathcal{A}_K is a subset of the product $\prod_{\nu} K_{\nu}$ of all completions of K for all equivalence classes ν of absolute value functions, consisting of sequences $(x_{\nu_1}, \dots, x_{\nu_n}, \dots)$ where $|x_{\nu_i}| \leq 1$ for all i except the finite number.

REMARK: Tikhonov’s theorem **implies that \mathcal{A}_K is locally compact.**

The strong approximation theorem

Strong approximation theorem: Consider the natural embedding $K \subset \mathcal{A}_K$. Then its image is a discrete, cocompact subgroup. Moreover, the projection of $\mathcal{A}_K \xrightarrow{P_{\nu_0}} \prod_{\nu \neq \nu_0} K_\nu$ to the product of all completions except one maps K to a dense subset of $R_{\nu_0}(\mathcal{A}_K)$.

REMARK: Further on, K is considered as a subring of \mathcal{A}_K .

Proof of Theorem 1. Step 1: Let $\mathcal{O}_{\mathcal{A}_K}$ be a ring of all integer adeles, that is, such $(x_{\nu_1}, \dots, x_{\nu_n}, \dots) \in \mathcal{A}_K$, that $|x_{\nu_i}| \leq 1$ for each non-archimedean absolute value. Then $\mathcal{O}_K = K \cap \mathcal{O}_{\mathcal{A}_K}$.

Step 2: Let now $P : \mathcal{A}_K \rightarrow \mathcal{A}_1$ be a projection of \mathcal{A}_K to the product of all completions except one archimedean. Since $\mathcal{O}_{\mathcal{A}_K}$ is open in \mathcal{A}_K , its projection to \mathcal{A}_1 is open in \mathcal{A}_1 (the projection is an open map).

Step 3: We obtain that the image $P(K) \cap P(\mathcal{O}_{\mathcal{A}_K})$ is dense in $P(\mathcal{O}_{\mathcal{A}_K})$. From Step 1, we obtain that $P(K) \cap P(\mathcal{O}_{\mathcal{A}_K})$ coincides with $P(\mathcal{O}_K)$.

Step 4: We proved that $P(\mathcal{O}_K)$ is dense in $\mathcal{A}_1 \cap P(\mathcal{O}_{\mathcal{A}_K})$. Therefore, its projection to the product of all archimedean completions except one is also dense.