Ergodic theory and symplectic packing

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Estruturas geométricas em variedades,

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Geometric structures

DEFINITION: "Geometric structure" on a manifold is a collection of tensors satisfying a certain set of differential equations.

Let me give some examples.

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

DEFINITION: Symplectic form on a manifold is a non-degenerate differential 2-form ω satisfying $d\omega = 0$.

Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put topology of uniform convergence with all derivatives on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

DEFINITION: The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of M. It acts on Teich by homeomorphisms.

DEFINITION: The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Today I will describe Teich and Γ in some interesting cases and explain some important concepts, such as **ergodicity of Γ-action.**

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$ as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: Teichmüller space of symplectic structures on M is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Define **the period map** $\mathsf{Per}: \mathsf{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\mathsf{Per}: \mathsf{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

The proof of Moser's theorem

THEOREM: (Moser)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

Proof. Step 1: We can locally find a section S for the Diff_0-action on Symp, producing a local decomposition $\operatorname{Symp} = O \times S$, where O is a Diff_0-orbit. Here O and S are both Frechet manifolds.

Step 2: The period map $P: S \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. Its fibers are submanifolds, hence locally path connected. By Theorem 1, the fibers of P are 0-dimensional. Therefore, P is locally a diffeomorphism. ∎

Non-Hausdorff points on symplectic Teichmüller space

Example of D. McDuff found in Salamon, Dietmar, *Uniqueness of symplectic structures*, Acta Math. Vietnam. 38 (2013), no. 1, 123-144.

Let $M = S^1 \times S^1 \times S^2 \times S^2$ with coordinates $\theta_1, \theta_2 \in S^1 \subset \mathbb{C}^*$ and $z_1, z_2 \in S^2$. Let $\varphi_{\theta,z}$ $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rotation around the axis $z \in \mathbb{C}P^1$ by the angle θ. **Consider the diffeomorphism $\Psi: M \longrightarrow M$ mapping $(\theta_1, \theta_2, z_1, z_2)$ to $(\theta_1, \theta_2, z_1, \varphi_{\theta_1, z_1}(z_2))$.**

THEOREM: Let ω_λ be the product symplectic form on $M = T^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ obtained as a product of symplectic forms of volume 1, 1, λ on T^2, $\mathbb{C}P^1$, $\mathbb{C}P^1$. **The form $\Psi^*(\omega_1)$ is homologous, but not diffeomorphic to ω_1.** However, **the form $\Psi^*(\omega_\lambda)$ is diffeomorphic to ω_λ for any $\lambda \neq 1$.**

(D. McDuff, *Examples of symplectic structures*, Invent. Math. 89 (1987), 13-36.)

Ergodic group action

DEFINITION: Let (M, μ) be a space with finite measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

CLAIM: A group G acts on M ergodically **if and only if any L^2-integrable G-invariant function on M is constant almost everywhere.**

Mapping class group action on $\operatorname{Teich}_s(A)$

DEFINITION: Symplectic volume of a symplectic manifold (M,ω), $\dim_{\mathbb{R}} M = 2n$, is $\int_M \omega^n$. Fix a positive number A, and let $\operatorname{Teich}_s(A)$ be the Teichmüller space of symplectic forms with symplectic volume A.

REMARK: The mapping class group $\frac{\operatorname{Diff}}{\operatorname{Diff}_0}$ acts on $H^2(M)$ and on $\operatorname{Teich}_s(A)$ **Quite often, this group is arithmetic, and this action is ergodic.**

In this case, **all semicontinuous symplectic invariants, evaluated on dense orbits, depend only on the symplectic volume.**

Known cases: K3 surface, hyperkähler manifolds, tori $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$, $n > 1$.

Kähler manifolds

DEFINITION: A Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: This is equivalent to $\nabla\omega = 0$, where ∇ is Levi-Civita connection.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection $Sp(n)$** if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Further on, we shall always assume that our hyperkähler manifolds have maximal holonomy.

Teichmüller space of symplectic structures for hyperkähler manifolds

DEFINITION: A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: (E. Amerik, V.) Let M be a maximal holonomy hyperkähler manifold. Then the period map $\mathsf{Per}: \mathsf{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes v such that $q(\omega,\omega) > 0$,** where q is a quadratic form on cohomology defined below.

REMARK: A similar result is proven for standard symplectic structures on a torus.

Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki) Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler (of maximal holonomy). Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**.

Remark: q **has signature** $(b_2-3,3)$**.** It is positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Ergodic group action

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of Γ on G/H is ergodic.**

Ratner's theorem

EXAMPLE: By Borel and Harish-Chandra theorem, **an integer lattice in a simple Lie group has finite covolume.**

DEFINITION: **Unipotent element** in a Lie group $G \subset GL(V)$ is an exponent of a nilpotent element in its Lie algebra.

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure of any Γ-orbit in G/H is an orbit of a Lie subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

EXAMPLE: Let V be a real vector space with integer lattice and a non-degenerate rational bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ the stabiliser of a positive vector $v \in V$, $H \cong SO^+(2, k)$, and $\Gamma \subset G$ an integer lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. **Then the closure of $\Gamma \cdot J$ in G/H is an orbit of a closed Lie subgroup $S \subset G$ containing H.** Moreover, **S is the smallest rational subgroup with this property.**

REMARK: In this situation, **either v is proportional to a rational vector, or $S = G$.** Indeed, there are no intermediate subgroups $SO^+(2, k) \subsetneq S \subsetneq SO^+(3, k)$.

Ergodicity of mapping class group action

THEOREM: (V., 2009)
Let M be a maximal holonomy hyperkähler manifold. **Then the image of the mapping class group Γ in $O(H^2(M,\mathbb{Z}))$ has finite index.**

COROLLARY: Γ acts on $\operatorname{Teich}_s(A)$ with dense orbits.

Proof: Applying Moore's theorem to Γ inside $G = SO(H^2(M,\mathbb{R}),q)$ and H the stabilizer of $\omega \in H^2(M,\mathbb{R})$, we obtain that the action of Γ on $\operatorname{Teich}_s(A) \subset H^2(M,\mathbb{R})$ **is ergodic on forms with fixed volume,** hence has dense orbits. ■

THEOREM: Let M be a hyperkähler manifold, Γ its mapping class group, and Teich_s the Teichmüller space of symplectic structures of hyperkähler type. **Then the dense orbits correspond to irrational symplectic classes,** and rational symplectic classes have closed orbits.

Proof: Follows from Ratner's theorems on classification of ergodic measures. ■

COROLLARY: On a hyperkähler manifold or a compact torus of dimension $2i > 2$, **any semicontinuous invariant of symplectic structures is constant on irrational symplectic forms** of standard type and fixed volume.

Gromov Capacity

DEFINITION: Let M be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii r, for all symplectic embeddings from a symplectic balls B_r to M.

DEFINITION: Define **symplectic volume** of a symplectic manifold (M, ω) as $\int_M \omega^{\frac{1}{2}\dim M}$.

REMARK: Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity r has volume $\geq \operatorname{Vol}(B_r)$. However, **there are manifolds of infinite volume with finite Gromov capacity.**

THEOREM: (Gromov)
Consider **a symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of C_r is r.

REMARK: This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He found the packing constant for 2 equal balls in $\mathbb{C}P^2$.

Ekeland-Hofer theorem

THEOREM: (Ekeland-Hofer)
Let M, N be symplectic manifolds, and $\varphi : M \longrightarrow N$ a diffeomorphism. Suppose that for all sufficiently small, convex open sets $U \subset M$, Gromov capacity satisfies $\mu(U) = \mu(\varphi(U))$. **Then φ is a symplectomorphism.**

REMARK: This can be used to define C^0- **(continuous) symplectomorphisms.**

REMARK: Ekeland-Hofer theorem implies a theorem of Gromov-Eliashberg: **symplectomorphism group is C^0-closed in the group of diffeomorphisms.**

Packing constants

DEFINITION: Let (K, ω_K) be a $2n$-dimensional symplectic manifold with finite volume, and (M, ω_M) a symplectic manifold. We assume that K admits a symplectic embedding to a bounded domain in $\mathbb{R}^{2n}$ with a flat symplectic structure. The corresponding **packing constant** is supremum of all ε such that $(K, \varepsilon\omega_K)$ admits a symplectic embedding to (M, ω_M). It is easy to see that **the packing constant is semicontinuous as a function of ω_M (Entov-V.)**

REMARK: Packing constant is a generalization of Gromov's symplectic capacity.

REMARK: Applying ergodicity to packing constants, we obtain that **these packing constant are universal,** that is, independent from the choice of an irrational symplectic structure as long as its volume stays constant. Indeed, the packing constants are semicontinuous as functions of ω, and any semicontinuous, MCG-invariant function is constant on dense orbits.

REMARK: Packing constants were computed explicitly when K is a union of symplectic balls, ellipsoids, and M is a torus or a hyperkähler manifold. In this situation, **the only obstruction to packing is the symplectic volume of M** (Entov-V.). For more exotic shapes, nothing is known, though everybody is sure that **for the hyperkähler manifolds and the tori the packing should be unobstructed, for any symplectic domain $K \subset \mathbb{R}^{2n}$**.