Hyperkähler manifolds with ergodic automorphism groups

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Egrodic group action on manifolds

DEFINITION: Let Γ be a group acting on a manifold M by measurable maps. We say that the action of Γ is **ergodic** if any Γ-invariant measurable subset of M is full measure or measure 0.

REMARK: Equivalently, (M, Γ) **is ergodic iff any Γ-invariant integrable function is constant almost everywhere.**

REMARK: From ergodicity it follows that **almost all orbits of Γ are dense, but converse is not true.**

Egrodic group action on K3

THEOREM: (Serge Cantat)
Let M be a K3-surface which is obtained as a degree (2,2,2)-hypersurface in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, and Γ its automorphism group. **Then Γ acts on M ergodically.**

REMARK: Since M has degree 2 in each variable, it has 2:1 projection to $\mathbb{C}P^1 \times \mathbb{C}P^1$. This gives an order 2 automorphism exchanging these two preimages. We obtain 3 order 2 automorphisms acting on M. "It is easy to see" (Cantat) that they generate the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

REMARK: $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is an index 6 subgroup in $PGL(2,\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \rtimes \Sigma_3$, where the projection to the symmetric group is given by $PGL(2,\mathbb{Z}) {\to} PGL(2,\mathbb{Z}/2) = \Sigma_3$ (Goldman, MacShane, Stantchev, 2015).

REMARK: The aim of this talk: **construct hyperkähler manifolds with ergodic automorphism groups.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple**, or **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

THEOREM: ("Bochner's vanishing")
Let M be a maximal holonomy hyperkähler manifold. **Then $H^{p,0} = 0$ for p odd, and $H^{p,0} = \mathbb{C}$ for p even.**

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = &\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ &-\frac{n-1}{2n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(3, b_2 - 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi: M \to X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the holomorphic symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **a holomorphic Lagrangian fibration**.

REMARK: The base of π **is conjectured to be rational.** Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

The hyperkähler SYZ conjecture

DEFINITION: A cohomology class $\eta \in H^{1,1}(M)$ is **nef** if it lies in the closure of the Kähler cone

A trivial observation: Let $\pi: M \to X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.**

DEFINITION: A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

The hyperkähler SYZ conjecture: Let L be a line bundle on a hyperkähler manifold, with $q(c_1(L), c_1(L)) = 0$, and $c_1(L)$ nef. Then L is semiample.

REMARK: The corresponding projective map $M \to \mathbb{P}(H^0(M, L)^*)$ is a Lagrangian fibration to its image, as follows from Matsushita theorem.

REMARK: Hyperkähler SYZ conjecture **is proven for all known examples of hyperkähler manifolds.**

The hyperbolic space and its isometries

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \ \mid \ q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Classification of automorphisms of hyperbolic space

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is a non-trivial oriented isometry acting on $V = \mathbb{R}^{1,n}$. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and a real eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$ (α is **"parabolic isometry"**).

REMARK: All eigenvalues of elliptic and parabolic isometries have absolute value 1. **Hyperbolic and elliptic isometries are semisimple** (that is, diagonalizable over $\mathbb{C}$), parabolic are not.

DEFINITION: The quadric $\{l \in \mathbb{P}V \ \ | \ \ q(l,l) = 0\}$ is called **the absolute**. It is realized as the boundary of the hyperbolic space $\mathbb{P}^+V$. Then **elliptic isometries have no fixed points on the absolute, parabolic isometries have 1 fixed point on the absolute, and hyperbolic isometries have 2**.

Automorphisms of hyperkahler manifolds

REMARK: Serge Cantat argues for a change of terminology to use **"loxodromic"** instead of "hyperbolic", and using "hyperbolic" for automorphisms which act trivially on a codimension 2 hyperspace.

REMARK: Let M be a hyperkähler manifold. Then the **BBF form has signature $(1, b_2 - 3)$ on $H^{1,1}(M)$.**

DEFINITION: An automorphism of a hyperkähler manifold (M, I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M, \mathbb{R})$.

REMARK: Let p be a parabolic automorphism of a hyperkähler manifold, and η its fixed point in $H^{1,1}(M)$ associated with the fixed point in the absolute. **Then η is proportional to an integer cohomology class which lies on the boundary of the Kähler cone.** Indeed, η can be obtained as a limit $p^i(w)$ for any Kähler class w on M.

Ergodic automorphism groups

Today's main result:

Theorem 1: Let M be a hyperkähler manifold admitting two parabolic automorphisms p_1, p_2 which have distinct fixed points on the absolute. Assume that SYZ conjecture holds for the corresponding two fixed points on the boundary of the Kähler cone. **Then p_1, p_2 generate a group acting on M ergodically.**

REMARK: If M admits a parabolic automorphism and $\operatorname{Aut}(M)$ is not virtually abelian, M admits parabolic automorphisms which have different fixed points on the absolute.

REMARK: In "Construction of automorphisms of hyperkähler manifolds" (E. Amerik, M. Verbitsky, 2017, Compositio) we proved that **any hyperkähler manifold with b_2 sufficiently big admits a projective deformation M_p with a parabolic automorphism.** Moreover, the automorphism group of M_p is arithmetic, hence M_p **admits parabolic automorphisms with different fixed points in the absolute.**

Parabolic automorphisms and Lagrangian fibrations

REMARK: The following theorem is due to Federico Lo Bianco, *"Dynamics of birational transformations of hyperkähler manifolds : invariant foliations and fibrations"*, Theorem B.

THEOREM: Let p be a parabolic automorphism of an algebraic hyperkähler manifold M, and $\pi: M \to X$ a Lagrangian fibration such that for a Kähler class ω on X, its pulback $\pi^*\omega$ is the class on the boundary of the Kähler cone fixed by p. **Then a certain power of p preserves the fibers of π.**

REMARK: In this case we say that p **preserves the Lagrangian fibration π.** Such a fibration **is unique,** because π is uniquely determined by the cohomology class $\pi^*\omega$, and p fixes one and only one point in the absolute.

Parabolic automorphisms and translations

REMARK: A complex torus T is not a priori a group, unless you fix the origin. However, its translation group, denoted by $\operatorname{Tr}(T)$, is a complex, commutative Lie group, and it is isomorphic to T as a manifold.

CLAIM: A translation τ_x of a torus T by a vector $x \in \operatorname{Tr}(T)$ has all its orbits dense if and only if x is not contained in a smaller torus $T' \subset \operatorname{Tr}(T)$. In this case, **$\tau_x$ is ergodic.**

Theorem 2: Let p be a parabolic automorphism of a hyperkähler manifold preserving a Lagrangian fibration $\pi: M \to X$. Then there exists a full measure, Baire second category subset $R \subset X$, such that for all $r \in R$ the fibers $\pi^{-1}(r)$ are tori, and **the automorphism p acts on $\pi^{-1}(r)$ with dense orbits.**

REMARK: Theorem 2 implies Theorem 1. Indeed, let Γ be the group generated by two parabolic automorphisms p_1, p_2, φ a Γ-invariant measurable function, and π_1, π_2 the Lagrangian fibrations associated with p_1, p_2. Since φ is p_i-invariant, it is constant almost everywhere on almost all fibers of π_i. However, a function which is constant on fibers of π_i is constant on M, because these fibers are transversal and complementary.

Hodge structures

DEFINITION: Let $V_{\mathbb{R}}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **integer or rational Hodge structure** if one fixes an integer or rational lattice $V_{\mathbb{Z}}$ or $V_{\mathbb{Q}}$ in $V_{\mathbb{R}}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is an inteber/rational map which is $U(1)$-invariant.

DEFINITION: Polarization on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \overline{x}) > 0 \qquad (*)$$

("Riemann-Hodge relations") for each non-zero $x \in V^{p,q}$.

Hodge structures of weight 1

DEFINITION: Two complex tori T_1, T_2 are called **isogeneous** if there exists a surjective finite holomorphic map $T_1 \to T_2$.

REMARK: The category of complex tori **is equivalent to the category of integer Hodge structures of weight (1,0) and (0,1)** (such Hodge structures are called **"Hodge structures of weight 1"**). The category of complex tori up to isogeny **is equivalent to the category of rational Hodge structures of weight 1.**

REMARK: Under this correspondence, **abelian manifolds correspond to Hodge structures admitting a polarization.**

CLAIM: The category $\mathcal{C}$ of rational Hodge structures admitting a polarization is semisimple, that is, any object of $\mathcal{C}$ is a direct sum of irreducible ones.

REMARK: In particular, **the category of abelian manifolds up to isogeny is semisimple.**

Variations of Hodge structures

DEFINITION: Let M be a complex manifold. A **variation of Hodge structures (VHS)** on M is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution and Hodge structures at each point, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy the following conditions:
(a) the polarization and the real structure are parallel with respect with ∇.
(b) **("Griffiths transversality condition")** $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$

CLAIM: Polarized integer variations of Hodge structures of weight 1 **are the same as holomorphic Abelian fibrations.**

EXAMPLE: Let $\pi : M \to X$ be a proper holomorphic surjective submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with the fiber in x the k-th cohomology of $\pi^{-1}(x)$, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. **This defines a variation of Hodge structures.**

Deligne's semisimplicity theorem

THEOREM: (Deligne's semisimplicity theorem, P. Deligne, "Un théorème de finitude pour la monodromie", 1984)
Let V be a polarized, rational variation of Hodge structures over a quasiprojective base M. Then the underlying flat bundle can be decomposed as $V = \bigoplus_i W_i \otimes L_i$, where L_i correspond to pairwise non-isomorphic irreducible representations of $\pi_1(M)$, and W_i are trivial representations. Moreover, W_i **are equipped with a Hodge structure,** L_i **are equipped with the structure of a VHS, and the decomposition** $V = \bigoplus_i W_i \otimes L_i$ **is compatible with the Hodge structures on** W_i, L_i**.**

Applying this result to abelian fibration, notice that one of two summands in $W_i \otimes L_i$ has weight 0, and another has weight 1. This gives a corollary:

COROLLARY: Let V be an abelian fibration (that is, a fibration with fiber an abelian variety, admitting a globally defined integer Kähler class). After passing to an isogeneous fibration V_1, **we can decompose** V_1 **onto a product of abelian fibrations with an irreducible monodromy of Gauss-Manin connection and an isotrivial fibration.**

AM-GM inequality and products of Hermitian forms

REMARK: Let $\alpha_1, ..., \alpha_n$ be positive numbers. Their **arithmetic mean** is $\frac{\sum \alpha_i}{n}$ and **geometric mean** is $\sqrt[n]{\prod_i \alpha_i}$. **AM-GM inequality** states that $\frac{\sum \alpha_i}{n} \geqslant \sqrt[n]{\prod_i \alpha_i}$ **and the equality happens if and only if all α_i are equal.**

COROLLARY: Let $\alpha_1, ..., \alpha_n$ be positive numbers such that $\sum \alpha_i = n$ and $\prod \alpha_i = 1$. Then all $\alpha_i = 1$.

Lemma 1: Let ω_1, ω_2 be Hermitian forms on a vector space $V = \mathbb{C}^n$, and h_1, h_2 the corresponding Hermitian forms. **Suppose that $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$. Then $\omega_1 = \omega_2$.**

Proof: Simultaneous diagonalization theorem implies that h_2 can be diagonalized in an orthonormal basis for h_1. Let $A = \omega_2 \omega_1^{-1}$ be the corresponding diagonal matrix. Clearly, $\frac{\omega_1 \wedge \omega_2^{n-1}}{\omega_2^n} = \frac{1}{n} \operatorname{Tr}(A)$ and $\frac{\omega_1^n}{\omega_2^n} = \det A$. By AM-GM inequality, $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$ implies that all eigenvalues of A are 1, and $\omega_1 = \omega_2$. ■

Lagrangian fibrations and Fujiki formula

Proposition 1: **(Voisin)** Let $\pi: M \to X$ be a Lagrangian fibration on a hyperkähler manifold. Then any smooth fiber $T := \pi^{-1}(x)$ is a torus, and, moreover, **the natural restriction map $H^2(M) \to H^2(T)$ has rank 1.**

Proof. Step 1: The normal bundle NT of T is trivial because $NT = \pi^* T_x X$, and its tangent bundle is trivial because it is dual to the normal bundle. **Then T is a torus,** because torus is the only compact Kähler manifold with trivial tangent bundle.

Step 2: Let $\omega_0 \in H^2(X)$ be a class on X satisfying $\int_X \omega_0^n = 1$, where $n = \dim_{\mathbb C} X$, and $\omega_1, \omega_2 \in H^2(M)$ any cohomology classes. We need to show that $\omega_1|_T$ is proportional to $\omega_2|_T$. Since T is Lagrangian, the (2,0)-forms are restricted to zero, and we may assume that $\omega_1, \omega_2 \in H^{1,1}(M)$. Picking ω_1 Kähler and replacing ω_2 by an appropriate linear combination, we may also assume that ω_2 is also Kähler. Clearly, $q(\pi^*\omega_0, \pi^*\omega_0) = 0$. By Fujiki formula, $\int_T \omega_i^n = \int_M \omega_i^n \wedge \pi^*\omega_0^n = q(\omega_i, \pi^*\omega_0)^n$ and

$$\int_T \omega_2 \wedge \omega_1^{n-1} = \int_M \omega_2 \wedge \omega_1^{n-1} \pi^*\omega_0^n = q(\omega_2, \pi^*\omega_0) q(\omega_1, \pi^*\omega_0)^{n-1}.$$

Rescaling ω_i in such a way that $q(\omega_1, \pi^*\omega_0) = q(\omega_2, \pi^*\omega_0) = 1$, we obtain that $\omega_1 \wedge \omega_2^{n-1} = \omega_1^n = \omega_2^n$ for two Kähler classes on a torus. By Lemma 1, this implies that ω_1 is cohomologous to ω_2. ■

Oguiso's theorem on Lagrangian fibrations

Corollary 1: (similar to K. Oguiso, "Picard number of the generic fiber of an abelian fibered hyperkaehler manifold", Theorem 1.1)
Let $\pi : M \to X$ be a Lagrangian fibration on a hyperkähler manifold, $X_0 \subset X$ a smooth locus of π, and V the variation of Hodge structures over X_0 associated with $H^1(\pi^{-1}(x))$, for all $x \in X_0$. Then **V is irreducible as a variation of Hodge structures.** Moreover, **the monodromy representation is irreducible, unless $\dim_{\mathbb C} M = 2$.**

The proof is implied by Voisin's theorem (if V is not irreducible, the symplectic form can be decomposed) and Deligne's invariant cycle theorem (all monodromy invariant vectors in V are obtained as restrictions of the cohomology classes in $H^*(M)$).

PROPOSITION: Let p be a parabolic automorphism of a hyperkähler manifold, preserving a Lagrangian fibration $\pi : M \to X$. For a regular value $x \in X$, let $p_x \in \operatorname{Tr}(\pi^{-1}(x))$ be the corresponding parallel transport of the torus $\pi^{-1}(x)$, and P_x its closure in $\operatorname{Tr}(\pi^{-1}(x))$. Assume that the dimension of P_x is maximal. **Then P_x is a monodromy-invariant subtorus $\operatorname{Tr}(\pi^{-1}(x))$.**

The proof (omitted) uses the semi-continuity of P_x as a function of $x \in X$.

Ergodicity of parabolic automorphisms

COROLLARY: Let $\pi : M \to X$ be a Lagrangian fibration, and p a parabolic automorphism. **Then p acts with dense orbits on a general fiber of π.**

Proof: The closure of an orbit of maximal dimension is given by a monodromy-invariant subspace in $H^1(\pi^{-1}(x))$. However, **the monodromy of the Gauss-Manin local system is irreducible by Corollary 1.** ■