Complex geometry and the isometries of the hyperbolic space

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Technion,
Geometry and Topology Seminar,
June 15, 2023

Joint work with E. Amerik

Kähler manifolds

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla: \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

REMARK: (the Hodge decomposition)
The second cohomology of a compact Kähler manifold are decomposed as $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, where $H^{2,0}(M)$ is the space of all cohomology classes which can be represented by holomorphic $(2, 0)$-forms, $H^{0,2}(M)$ its complex conjugate, and $H^{1,1}(M)$ the classes which can be represented by I-invariant forms.

Kummer surfaces

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with a non-degenerate, holomorphic $(2,0)$-form.

EXAMPLE: For any complex manifold M, **the total space T^*M of the cotangent bundle is holomorphically symplectic.**

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity $\mathbb{C}^2/\pm 1$.**

REMARK: Let M be a 2-dimensional complex manifold which is holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

DEFINITION: Take a 2-dimensional complex torus T, then all 16 singular points of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

DEFINITION: A K3 surface is a complex deformation of a Kummer surface.

K3 surfaces

"K3: Kummer, Kähler, Kodaira" (the name is due to A. Weil).

"Faichan Kangri is the 12th highest mountain on Earth."

Topology of K3 surfaces

THEOREM: Any complex compact surface with $c_1(M) = 0$ and $H^1(M) = 0$ **is isomorphic to K3.** Moreover, **it is Kähler.**

CLAIM: 1. $\pi_1(K3) = 0$,

2. The second homology and cohomology of K3 is torsion-free.

3. $b_2(K3) = 22$, and the signature of its intersection form is $(3, 19)$.

4. The intersection form of K3 is even, and the corresponding quadratic lattice is $U^3 \oplus (-E_8)^2$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the Coxeter matrix for the group E_8.

Complex surfaces and hyperbolic lattices

REMARK: Let M be a complex surface of Kähler type. **Then the signature of the intersection form on $H^{1,1}(M)$ is $(1, h^{1,1} - 1)$.**

THEOREM: Let M be a projective K3 surface, and $\operatorname{Aut}(M)$ its group of complex automorphisms. **Then the natural map $\operatorname{Aut}(M) \longrightarrow O(H^{1,1}(M))$ has finite kernel.**

REMARK: Since $H^{1,1}(M)$ has signature $(1, h^{1,1}-1)$, the group $O^+(H^{1,1}(M))$ is the group of isometries of a hyperbolic space $\mathbb{P}H^{1,1}(M)$. If we are interested in dynamics, the "finite kernel" does not make any difference. **The automorphisms of M can be classified in the same way as isometries of the hyperbolic space of constant sectional curvature.**

Classification of automorphisms of a hyperbolic space

REMARK: The group $O(m,n), m,n > 0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \ \mid \ q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is an isometry acting on V. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic (or loxodromic) isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$. (α is **"parabolic isometry"**)

DEFINITION: An automorphism of a K3 surface (M,I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M,\mathbb{R})$.

Classification of automorphisms of complex surfaces

CLAIM: An elliptic automorphism of K3 has finite order.

Proof: Any elliptic isometry of $\mathbb{H}^n$ which has infinite orbit generates a group which is dense in a compact torus, and $O(H^{1,1}(M,\mathbb{Z}))$ is discrete. ■

REMARK: Recall that **an elliptic surface** is a complex surface equipped with a holomorphic, surjective map $M \longrightarrow C$ to a curve, with general fibers elliptic curves.

THEOREM: (M. Gizatullin) Let τ be a parabolic automorphism of a complex surface M. **Then M is elliptic, and a finite power of τ preserves the fibers of elliptic projection.**

THEOREM: (S. Cantat) Let τ be a parabolic automorphism of a surface preserving an elliptic fibration $\pi: M \longrightarrow C$. **Then the action of τ has dense orbits on almost all fibers of π.**

Egrodic group action on manifolds

DEFINITION: Let Γ be a group acting on a manifold M by measurable maps. We say that the action of Γ is **ergodic** if any Γ-invariant measurable subset of M is full measure or measure 0.

REMARK: Equivalently, (M,Γ) **is ergodic iff any Γ-invariant integrable function is constant almost everywhere.**

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

REMARK: From ergodicity it follows that **almost all orbits of Γ are dense,** but **converse is not true.**

Egrodic group action on K3

THEOREM: (Serge Cantat)
Let M be a K3-surface which is obtained as a degree (2,2,2)-hypersurface in $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, and Γ its automorphism group. **Then Γ acts on M ergodically.**

REMARK: Since M has degree 2 in each variable, it has 2:1 projection to $\mathbb{C}P^1 \times \mathbb{C}P^1$. This gives an order 2 automorphism exchanging these two preimages. We obtain 3 order 2 automorphisms acting on M. They generate the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

REMARK: $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ **is an index 6 subgroup in** $PGL(2, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \rtimes \Sigma_3$**,** where the projection to the symmetric group is given by $PGL(2, \mathbb{Z}) \longrightarrow PGL(2, \mathbb{Z}/2) = \Sigma_3$ (Goldman, MacShane, Stantchev, 2015).

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{2n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n\right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi : M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the holomorphic symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **a holomorphic Lagrangian fibration**.

REMARK: The base of π **is conjectured to be rational.** Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

The hyperkähler SYZ conjecture

DEFINITION: A cohomology class $\eta \in H^{1,1}(M)$ is **nef** if it lies in the closure of the Kähler cone

A trivial observation: Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.**

DEFINITION: A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

The hyperkähler SYZ conjecture: Let L be a line bundle on a hyperkähler manifold, with $q(c_1(L), c_1(L)) = 0$, and $c_1(L)$ nef. Then L is semiample.

REMARK: The corresponding projective map $M \longrightarrow \mathbb{P}(H^0(M, L)^*)$ is a Lagrangian fibration to its image, as follows from Matsushita theorem.

REMARK: Hyperkähler SYZ conjecture **is proven for all known examples of hyperkähler manifolds.**

Automorphisms of hyperkahler manifolds

REMARK: Let M be a hyperkähler manifold. Then the **BBF form has signature $(1, b_2-3)$ on $H^{1,1}(M)$.** An automorphism of a hyperkähler manifold (M, I) is called **elliptic (parabolic, hyperbolic)** if it is elliptic (parabolic, hyperbolic) on $H^{1,1}_I(M, \mathbb{R})$.

REMARK: Let p be a parabolic automorphism of a hyperkähler manifold, and η its fixed point in $H^{1,1}(M)$ associated with the fixed point in the absolute. **Then η is proportional to an integer cohomology class which lies on the boundary of the Kähler cone.** Indeed, η can be obtained as a limit $p^i(w)$ for any Kähler class w on M.

THEOREM: (Federico Lo Bianco)
Let p be a parabolic automorphism of an algebraic hyperkähler manifold M, and $\pi: M \longrightarrow X$ a Lagrangian fibration such that for a Kähler class ω on X, its pulback $\pi^*\omega$ is the class on the boundary of the Kähler cone fixed by p. **Then a certain power of p preserves the fibers of π.**

REMARK: In this case we say that p **preserves the Lagrangian fibration π.** If SYZ comjecture holds for M, **a power of any parabolic automorphism preserves a Lagrangian fibration.** Such a fibration **is unique,** because π is uniquely determined by the cohomology class $\pi^*\omega$, and p fixes one and only one point in the absolute.

Ergodic automorphism groups

Today's main result:

Theorem 1: Let M be a hyperkähler manifold admitting two parabolic automorphisms p_1, p_2 which have distinct fixed points on the absolute. Assume that SYZ conjecture holds for the corresponding two fixed points on the boundary of the Kähler cone. **Then p_1, p_2 generate a group acting on M ergodically.**

REMARK: If M admits a parabolic automorphism and $\operatorname{Aut}(M)$ is not virtually abelian, M admits parabolic automorphisms which have different fixed points on the absolute.

REMARK: In "Construction of automorphisms of hyperkähler manifolds" (E. Amerik, M. V., 2017, Compositio) we proved that **any hyperkähler manifold with b_2 sufficiently big admits a projective deformation M_p with a parabolic automorphism.** Moreover, the automorphism group of M_p is arithmetic, hence M_p **admits parabolic automorphisms with different fixed points in the absolute.**

Parabolic automorphisms and translations

REMARK: A complex torus T is not a priori a group, unless you fix the origin. However, its translation group, denoted by $\operatorname{Tr}(T)$, is a complex, commutative Lie group, and it is isomorphic to T as a manifold.

CLAIM: A translation τ_x of a torus T by a vector $x \in \operatorname{Tr}(T)$ has all its orbits dense if and only if x is not contained in a smaller torus $T' \subset \operatorname{Tr}(T)$. In this case, **$\tau_x$ is ergodic.**

Theorem 2: Let p be a parabolic automorphism of a hyperkähler manifold preserving a Lagrangian fibration $\pi: M \longrightarrow X$. Then there exists a full measure, Baire second category subset $R \subset X$, such that for all $r \in R$ the fibers $\pi^{-1}(r)$ are tori, and **the automorphism p acts on $\pi^{-1}(r)$ with dense orbits.**

REMARK: **Theorem 2 implies Theorem 1.** Indeed, let Γ be the group generated by two parabolic automorphisms p_1, p_2, φ a Γ-invariant measurable function, and π_1, π_2 the Lagrangian fibrations associated with p_1, p_2. Since φ is p_i-invariant, it is constant almost everywhere on almost all fibers of π_i. However, a function which is constant on fibers of π_i is constant on M, because these fibers are transversal and complementary.