Perverse coherent sheaves on hyperkähler manifolds and Weil conjectures

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IMPA, Rio de Janeiro

Brazil-China Joint Mathematical Meeting, July 17th - 21st, 2023

July 20, 2021, Foz de Iguacu

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K: TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **hyperkähler**.

CLAIM: A hyperkähler manifold (M,I,J,K) is **holomorphically symplectic** (equipped with a holomorphic, non-degenerate 2-form). Recall that M is equipped with 3 symplectic forms ω_I, ω_J, ω_K.

LEMMA: The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on (M,I).** ■

Converse is also true (for compact manifolds of Kähler type).

THEOREM: (Calabi-Yau) Let M be a compact, holomorphically symplectic Kähler manifold. Then **M admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Induced complex structures

LEMMA: Let ∇ be a torsion-free connection on a manifold, $I \in \operatorname{End} TM$ an almost complex structure, $\nabla I = 0$. **Then I is integrable.**

Proof: Let $X, Y \in T^{1,0}M$, then $[X,Y] = \nabla_X Y - \nabla_Y X \in T^{1,0}M$. ■

DEFINITION: **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors. This family of complex structures is also called **the twistor family**.

REMARK: Because of the Lemma above, **induced complex structure operators are integrable.**

$SU(2)$-action on cogomology

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

COROLLARY: The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$ acts on cohomology of a compact Kähler manifold, giving **the Hodge decomposition:** $H^*(M) = \bigoplus H^{p,q}(M)$.

REMARK: **The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition**, for $U(1)$-action defined by $\rho(t) = e^{tW}$.

THEOREM: (Bott)
Let M be a Riemannian manifold, and $V: TM \longrightarrow TM$ an endomorphism satisfying $\nabla V = 0$. **Then $[V, \Delta] = 0$.** In particular, if M is compact, V acts on cohomology of M.

COROLLARY: For any hyperkähler manifold, the group $SU(2)$ of unitary quaternions **defines $SU(2)$-action on cohomology**.

REMARK: For each induced complex structure L, we have an embedding $U(1) \subset SU(2)$. Therefore, **the Hodge decomposition for $L = aI + bJ + cK$ is induced by the $SU(2)$-action.**

Trianalytic subvarieties

DEFINITION: **Trianalytic subvarieties** are closed subsets which are complex analytic with respect to I, J, K.

REMARK: **Trianalytic subvarieties are hyperkähler submanifolds outside of their singularities.**

REMARK: Let $[Z]$ be a fundamental class of a compact complex subvariety Z on a Kähler manifold. **Then $[Z]$ is $U(1)$-invariant.**

COROLLARY: **A fundamental class of a trianalytic subvariety is $SU(2)$-invariant.**

THEOREM: Let (M, I, J, K, g) be a hyperkähler manifold, and $Z \subset (M, I)$ a complex analytic subvariety. Assume that the fundamental class of Z is $SU(2)$-invariant. **Then Z is trianalytic.**

COROLLARY: Let M be a hyperkähler manifold. Then there exists a countable subset $S \subset \mathbb{C}P^1$, such that for any induced complex structure $L \in \mathbb{C}P^1 \backslash S$, **all compact complex subvarieties of (M, L) are trianalytic.**

COROLLARY: For M compact and hyperkähler, and $L \in \mathbb{C}P^1$ generic, **the manifold (M, L) has no complex divisors.** In particular, **it is non-algebraic.**

Hyperholomorphic bundles

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda^{1,1}_I(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions. **All Chern classes of bundles admitting hyperholomorphic connections are $SU(2)$-invariant.** Indeed, $\Lambda^{2p}(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda^{p,p}_I(M)$.

THEOREM: Let B be a stable holomorphic bundle on (M, I), where (M, I, J, K) is hyperkähler. Then the (unique) **Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.**

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$-invariant $c_1(B)$ and $c_2(B)$ **is a hyperkähler variety** (possibly singular).

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure (that is, a complex structure outside of a certain countable set). **Then any stable bundle on (M, L) is hyperholomorphic.**

Bando and Siu: admissible connections with singularities

DEFINITION: Let M be Kähler and $Z \subset M$ a closed subset of Hausdorff codimension $\geqslant 4$. A Chern connection ∇ on a bundle B is called **admissible** if the form $\operatorname{Tr}(\Theta_B \wedge \Theta_B)$ is integrable, and $|\operatorname{Tr}\Theta_B|$ is bounded.

THEOREM: (Bando, Siu)
The bundle B on $M\backslash Z$ **can be extended to a coherent sheaf on M if and only if it admits an admissible connection.**

THEOREM: (Bando, Siu)
A torsion-free coherent sheaf on M **is polystable if and only if its non-singular part admits an admissible connection with $\Lambda\Theta_B = const \cdot \operatorname{Id}_B$.**

Hyperholomorphic sheaves

DEFINITION: A reflexive coherent sheaf is called **stable hyperholomorphic** if it admits a hyperholomorphic connection outside of its singular set.

THEOREM: A stable hyperholomorphic sheaf on a hyperkähler manifold is stable. Moreover, any stable reflexive coherent sheaf with $c_1(F)$, $c_2(F)$ $SU(2)$-invariant **admits a unique admissible hyperholomorphic connection.**

DEFINITION: Define **the category of hyperholomorphic sheaves** as the full subcategory of the category of coherent sheaves with objects obtained as extensions and subquotients of stable hyperholomorphic sheaves on trianalytic subvarieties.

THEOREM: Let I be a generic complex structure on a compact hyperkähler manifold. **Then all coherent sheaves on (M, I) are hyperholomorphic.**

THEOREM: Let M be a hyperkähler complex torus or a Hilbert scheme of K3, and I, I' two complex structures. **Then the categories of hyperholomorphic sheaves on (M, I) and (M, I') are equivalent.**

REMARK: **Conjecturally, this is always true.**

Perverse coherent sheaves: an overview

DEFINITION: Denote by $\Gamma_c(U, F)$ the space of compactly supported sections of a sheaf F on U. Then $U \longrightarrow \Gamma_c(U, F)^*$ is also sheaf. The corresponding derived functor is called **the functor of Verdier duality**. It gives Poincare duality for locally constant sheaves and Serre's duality for coherent sheaves (in classical topology).

DEFINITION: Let's denote by $D^b \mathcal{C}oh(M)$ **the derived category of complexes of sheaves of $\mathcal{O}_M$-modules with coherent cohomology,** only finitely many of which are non-zero.

I shall define **category of perverse sheaves** as the heart of a certain Verdier-invariant t-structure on derived category of $D^b \mathcal{C}oh(M)$, where M is a hyperkähler manifold with a general complex structure. In other words, perverse coherent sheaves are a solution to the following problem.

Problem: Find a full abelian sub-category P in $D^b \mathcal{C}oh(M)$ which is Serre self-dual and satisfies $D^b(P) = D^b \mathcal{C}oh(M)$.

REMARK: If we replace $D^b \mathcal{C}oh(M)$ by derived category of constructible sheaves, the solution is given by the (usual) perverse sheaves.

Verdier duality and perversity

DEFINITION: Let $D(\leqslant k)$ the the triangulated category of constructible sheaves on a complex manifold M with support in complex subvarieties of dimension $\leqslant k$. **It is easy to see that $D(\leqslant k-1) \subset D(\leqslant k)$ is a thick subcategory.**

DEFINITION: Consider the set $Z(k)$ of all smooth k-dimensional complex subvarieties (not necessarily closed) in M, defined up to excision of positive codimensional subsets, with the morphisms given by open embeddings. **Let $\mathcal{C}(k)$ be the category of all locally constant sheaves on $Z \in Z(k)$, with the morphisms defined on Zariski open, dense subsets.**

CLAIM: The quotient category $D(k) := D(\leqslant k-1)/D(\leqslant k)$ is obtained as a derived category for $\mathcal{C}(k)$, and **Verdier duality acts on $D(k)$ by sending $\mathcal{C}(k) \subset D(k)$ to $\mathcal{C}(k)[2k] \subset D(k)$.**

Proof: This is clear, because all objects of $D(k)$ are local systems on manifolds of real dimension $2k$, and Verdier duality puts a local system on an oriented $2k$-dimensional manifold to its dual shifted by $2k$. ■

Perverse sheaves

DEFINITION: **Perversity** is a map $v: \{0,1,2,...,n\} \longrightarrow \mathbb{Z}$.

DEFINITION: Fix a perversity v. Consider new t-structures on each $D(k)$ obtained by shifting the standard t-structures on $D(k)$ by $v(k)$ (that is, replacing $D(k)^{\leqslant i}$ by $D(k)^{\leqslant i+v(k)}$ for each $D_L(k)$). **Perverse t-structure** on D is a t-structure obtained by gluing these t-structures.

DEFINITION: **Middle perversity** is one which maps k to $-k$.

THEOREM: Let $\tau_{\leqslant i}$ be the t-structure on D (derived category with constructible cohomology) associated with self-dual perversity as above, and V the Verdier duality functor. Then $V(\tau_{\leqslant i}) = \tau_{\geqslant -i}$. In particular, **the heart of τ is Verdier self-dual.**

Proof: On each $D(k)$, Verdier duality moves complexes with cohomology in degree $\geqslant i$ to complexes with cohomology in degree $\leqslant 2k-i$. The middle perversity $D(k)^{\geqslant r}$ is complexes with cohomology in degree $\geqslant r+k$. This is shifted to complexes with cohomology in degree $\leqslant 2k-r-k = k-r$. This is precisely $D(k)^{\leqslant -r}$. ■

Middle perversity for coherent sheaves

Problem: The 6 functors construction works satisfactory for coherent sheaves, if we replace D_L with stratified coherent sheaves (ones which are non-singular on each open stratum). Then Verdier duality shifts perversities by mapping $v: \{0,1,2,...,n\} \longrightarrow \mathbb{Z}$. to $v^*(t) = v(t) - t$. **This gives the middle perversity $v = -t/2$, which makes no sense unless all t are even.** Generally speaking, **there is no middle perversity for coherent sheaves.**

Solution: **There is one in hyperkähler geometry!** Indeed, for a general manifold in the twistor family, all subvarieties are trianalytic, hence even-dimensional.

Coherent perverse sheaves on hyperkähler manifolds

DEFINITION: Let I be a general complex structure in the twistor family on a hyperkähler manifold M. Define **coherent perverse sheaves** on (M, I) as perverse coherent sheaves obtained from the self-dual (middle) perversity $v(t) = -t/2$. This makes sense, because all complex subvarieties of (M, I) are even-dimensional.

REMARK: Perverse sheaves form an Artinian and Noetherian category (that is, all descending and ascending chains of subsheaves terminate). Indeed, it is Noetherian, because rank of a sheaf is bounded, but then by duality it is Artinian. Thus it makes sense to speak of **"irreducible perverse sheaves"** (ones without sub-objects).

REMARK: All irreducible perverse sheaves are obtained as "IC-extensions" of stable sheaves on trianalytic subvarieties, in the same way as it is done with the usual IC-extensions.

REMARK: The deformation spaces of irreducible perverse sheaves are "better behaved" than deformation spaces of coherent sheaves: they are expected to be smooth and hyperähler. **This is proven only for small dimensions and for bundles.**

Weights of Frobenius action

Fix primes $l \neq p$, and let $q = p^r$. The $\mathbb{Z}_l$-constructible sheaves are obtained as inverse limits of $\mathbb{Z}/l^n$-constructible sheaves in etale category. In BBD (Beilinson-Bernshtein-Deligne) setup we are working with constructible l-adic sheaves of vector spaces over $\mathbb{Q}_l$, obtained from constructible $\mathbb{Z}_l$-sheaves by tensoring with $\mathbb{Q}$.

DEFINITION: An element $\alpha \in \mathbb{Q}_l$ is called **pure of q-weight d** if it is algebraic and all its Galois conjugate have absolute value $q^{d/2}$, and **mixed of q-weight d** if they have absolute values $q^{(d-i)/2}$, $i \geqslant 0$. We say that an endomorphism of a vector space is **pure/mixed of weight d** if all its eigenvalues are pure/mixed.

DEFINITION: Let M be a variety over $\mathbb{F}_q$, and $\mathcal{A}_{\mathbb{Q}_l}(M)$ the category of l-adic constructible sheaves (in etale topology). The geometric Frobenius is etale, hence it acts on fibers of any sheaf $F \in \mathcal{A}_{\mathbb{Q}_l}(M)$. A sheaf F is **pure (mixed) of weight d** if for all $i \geqslant r$, the Frobenius Fr^r acts on the fiber of F at each $x \in M(\mathbb{F}_q)$ with pure p^r-weight d (mixed weight $\leqslant d$).

Weil conjectures and Frobenius action

THEOREM: (BBD version of Weil conjectures)
Let F be a weight $\leqslant d$ mixed l-adic constructible sheaf on a variety of finite type M over $\mathbb{F}_q$. **Then** Fr **acts on the etale cohomology $H_c^k(F)$ with compact support with mixed weights $\leqslant d+k$.**

DEFINITION: Pure/mixed complexes of weight d are complexes of sheaves such that the Frobenius acts on its p-th cohomogy with pure/mixed weight $d+p$.

COROLLARY: Let $f: X \longrightarrow Y$ be a proper morphism of varieties over $\mathbb{F}_q$, and F a pure sheaf on X. **Then the derived pushforward $R\pi_*(F)$ is pure of the same weight.**

THEOREM: (Purity theorem) A pure perverse sheaf is a direct sum of simple (that is, indecomposable) perverse sheaves. In other words, **any exact sequence of pure perverse sheaves of the same weight splits.**

Back to hyperkähler geometry: twistor space

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I^{=}_{\mathrm{Tw}} - \mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

REMARK: Rational lines $\mathbb{C}P^1 \times \{x\} \subset \mathrm{Tw}(M)$ are called **horizontal $\mathbb{C}P^1$'s**.

REMARK: For M compact, $\mathrm{Tw}(M)$ never admits a Kähler structure.

Twistor lifting

DEFINITION: Twistor lifting of a coherent sheaf F on (M,I) is a sheaf $\mathcal{F}$ on $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ which is flat over each horizontal $\mathbb{C}P^1$:

$$\operatorname{Tor}^i(\mathcal{F}, \mathcal{O}_{\mathbb{C}P^1\times\{x\}}) = 0 \quad \forall i > 0$$

and equipped with an isomorphism $\mathcal{F}\Big|_{(M,I)} = F$.

REMARK: There is a triangulated category of twistor liftings, and it is possible to define the corresponding category of perverse sheaves.

REMARK: "Twistor lifting" of F **is roughly equivalent to a choice of connection (not necessarily Hermitian) on F over M with $SU(2)$-invariant curvature.**

DEFINITION: A twistor lifting $\mathcal{F}$ is **mixed/pure of weight $\leqslant d$ at $x \in M$** if $\mathcal{F}\Big|_{\mathbb{C}P^1\times\{x\}} \cong \oplus\mathcal{O}(k_i)$, with $k_i \leqslant d$ or $k_i = d$.

Weil conjectures for perverse coherent sheaves

Given a hyperkähler morphism (that is, a map whose graph is trianalytic) $f: M \longrightarrow M_1$, we obtain a derived pushforward of a sheaf equipped with a twistor lifting. **The following results are known only in smooth situation.** They are analogues of Weil conjecture and the purity theorem.

THEOREM: A derived direct image of a pure complex of twistor liftings is pure of the same weight.

THEOREM: A pure complex of perverse twistor liftings is a direct sum of simple perverse twistor liftings.