Stable bundles on positive elliptic fibrations

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Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n-manifold is called **Gauduchon** if $dd^c\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique, up to a constant multiplier.

REMARK: If ω is Gauduchon, then (by Stokes' theorem) $\int_M \omega^{n-1} dd^c f = 0$ for any f. The curvature Θ_L of a holomorphic line bundle L is well-defined up to $dd^c \log |h|$, where h is a conformal factor. Therefore, **for any line bundle L, the number $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$ is well defined.**

REMARK: Unlike the Kähler case, $\deg_\omega L$ is a holomorphic invariant of L, and **not topological.**

DEFINITION: Given a torsion-free coferent sheaf F of rank r, let $\det F := \Lambda^r F^{**}$. From algebraic geometry it is known that $\det F$ is a line bundle. Define **the degree** $\deg_\omega F := \deg_\omega \det F = \int_M \operatorname{Tr} \Theta_F \wedge \omega^{n-1}$.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional Gauduchon manifold (M,ω), and $\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$. A torsion-free sheaf F is called **stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$, where Θ_B is its curvature.

THEOREM: (**Kobayashi-Hitchin correspondence**; Donaldson, Buchsdahl, Uhlenbeck-Yau, Li-Yau, Lübke-Teleman): Let B be a holomorphic vector bundle. **Then B admits a Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

REMARK: This result **was generalized to coherent sheaves** by Bando and Siu.

REMARK: **Stability is required** if you want to classify vector bundles or construct their moduli spaces.

Positivity for stable bundles

"Bogomolov's inequality": if $\deg B = 0$ and B is Yang-Mills, then $\mathrm{Tr}(\Theta_B \wedge \Theta_B) \wedge \omega^{n-2}$ is a positive volume form, vanishing only in the points where the curvature Θ_B of B vanishes. I will explain its proof in the next slide.

DEFINITION: Let $r := \operatorname{rk} B$ and $\Delta(B) := 2rc_2(B) - (r-1)c_1^2(B)$. This cohomology class is called **the Bogomolov-Gieseker discriminant** of B.

REMARK: The form $\mathrm{Tr}(\Theta_B \wedge \Theta_B)$ is clearly closed. **Its cohomology class is equal to $const \cdot \Delta(B)$.**

COROLLARY: A stable bundle B on a Kähler manifold M with $c_1(B) = 0, c_2(B) = 0$ is flat.

Today I will give a version of this statement on manifolds equipped with foliations, in particular, **when M is equipped with a positive elliptic fibration.**

The Bogomolov-Lübke inequality

THEOREM: Let B be a stable bundle over a compact Kähler manifold M, ∇ its Yang-Mills connection, and θ its curvature. Assume that slope$(B) = 0$. Denote by $\Delta \in H^4(M)$ the discriminant of B, $\Delta = 2rc_2(B) - (r-1)c_1(B)^2$. **Then $\int_M \Delta \wedge \omega^{n-2} \geqslant 0$, and the inequality may happen only when $\Theta = 0$.**

Proof. Step 1: Since the connection ∇ is Hermitian, it preserves the natural real structure in $\Lambda^{1,1}(M) \otimes \operatorname{End}(B)$, $\eta \otimes b \longrightarrow \overline{\eta} \otimes b^\perp$, where by $b^\perp$ one understands the Hermitian adjoint endomorphism. Therefore, **we may assume that Θ is real with respect to this real structure.**

Step 2: Let $\theta_1, ..., \theta_n$ be an orthonormal basis in $\Lambda^{1,0}(M)$. Consider the decomposition $\Theta = \sum_{i \neq j} (\theta_i \wedge \overline{\theta}_j - \overline{\theta}_i \wedge \theta_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \overline{\theta}_i) \otimes a_i$ with $b_{ij}, a_i \in \mathfrak{u}(B)$. Let $\Xi := \operatorname{Tr}(\Theta \wedge \Theta)$. Then $\Xi \wedge \omega^{n-2} = \mathit{const} \operatorname{Tr}\left(-\sum_{i \neq j} b_{ij}^2 + \sum_{i \neq j} a_i a_j\right)$, where const is a positive constant. On the other hand, $\Lambda(\Theta) = \sum a_i = 0$, which brings $\sum_{i<j} a_i a_j = -\sum_i a_i^2$. This gives

$$\Xi \wedge \omega^{n-2} = \mathit{const} \operatorname{Tr}\left(-\sum_{i \neq j} b_{ij}^2 - \sum_i a_i^2\right).$$

Since the Killing form on $\mathfrak{u}(B)$ is negative definite, this sum is positive, and strictly positive unless $\Xi = 0$. ■

Transversally Kähler foliations

DEFINITION: Semi-Hermitian form is a form $\omega \in \Lambda^{1,1}(M,\mathbb{R})$ such that $\omega(Ix,x) \geqslant 0$ for any $x \in TM$ **(the inequality is strict iff ω is Hermitian)**.

DEFINITION: A **foliation** on a complex manifold M is a complex sub-bundle $F \subset TM$, $\dim_R F = 2$, closed under commutator (usually it is assumed to be holomorphic). A foliation is called **transversally Kähler** if M is equipped with a closed semi-Hermitian form ω_0 such that $\omega_0(x,\cdot) = 0$ for any $x \in F$ and ω_0 is Hermitian on TM/F.

REMARK: On a compact Kähler n-manifold (M,ω), **a semi-Hermitian form ω_0 is never exact.** Indeed, $\int_M \omega_0 \wedge \omega^{n-1} > 0$, hence ω_0 cannot be exact. On compact, complex, non-Kähler manifolds, **transversally Kähler foliations with exact ω_0 are quite common.**

EXAMPLE: The classical Hopf surface is $H := \mathbb{C}^2\backslash 0/\mathbb{Z}$, where $\mathbb{Z}$ acts as a multiplication by a complex number λ, $|\lambda| > 1$. Clearly, H is diffeomorphic to $S^1 \times S^3$, and fibered over $\mathbb{C}P^1$ with fiber $\mathbb{C}^*/\langle\lambda\rangle$.

CLAIM: Let $\pi: H \longrightarrow \mathbb{C}P^1$ be the standard projection, and $\omega_0 := \pi^*\omega_{\mathbb{C}P^1}$ be a pullback of the Fubini-Study form. Clearly, **ω_0 is exact, because $H^2(H) = 0$** (by Künneth formula). Therefore, **H admits a transversally Kähler, exact form.**

Prinipal elliptic fibrations.

DEFINITION: **A principal elliptic fibration** M is a complex manifold equipped with a free holomorphic action of a 1-dimensional compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

REMARK: It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

DEFINITION: Let $M \xrightarrow{\pi} X$ be a principal elliptic fibration, M compact. We say that M is **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact. **("Kähler class" is a cohomology class of a Kähler form).**

EXAMPLE: The classical Hopf surface introduced earlier.

EXAMPLE: A more general example is given by $\operatorname{Tot}(L^*)/\langle\mathbb{Z}\rangle$, where L is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because $\pi^*(c_1(L)) = 0$, and $c_1(L)$ is a Kähler class.

Calabi-Eckmann manifolds

Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$.

Now, let $M := (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. Its total space M is called **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$.

REMARK: The map $M \longrightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ **is a principal elliptic fibration**.

REMARK: The pullback of a Kähler form from $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ to M **is exact**, because $H^2(M) = 0$ (by Künneth formula).

Irregular and quasi-regular foliations

DEFINITION: A foliation is called **quasi-regular** if all its leaves are compact. If this is not so, it is called **irregular**. A foliation is called **regular** if all its leaves are compact, and the leaf space is smooth.

REMARK: The examples given above (Vaisman, Calabi-Eckmann) **are deformed naturally to irrregular foliated transversally Kähler manifolds.**

REMARK: Calabi-Eckmann manifolds were generalized by Lopez de Medrano, Verjovsky and Meersseman. The complex structure on Calabi-Eckmann can be deformed together with the foliation, giving **a transversally Kähler manifold with a foliation having non-compact leaves ("LVM-manifolds")**. A version of this construction is known as **the moment-angle manifolds**.

Subvarieties in manifolds equipped with positive elliptic fibrations

THEOREM: Let (M, Σ, ω_0) be a manifold equipped with a rank 1 holomorphic foliation Σ and an exact trasversally Kähler form ω_0, and $Z \subset M$ a complex subvariety. **Then Z is tangent to leaves of Σ everywhere.** If, in addition, Σ is quasi-regular, **then $Z = \pi^{-1}(Z_0)$,** where $Z_0 \subset M_0$ is a complex subvariety of the leaf space $M_0 = M/\Sigma$.

Proof: Let $k := \dim Z$. Since ω_0 is exact, we have $\int_Z \omega_0^k = 0$. Therefore, the restriction $\omega_0|_Z$ cannot be strictly positive; in other words, for each $z \in Z$ the tangent space $T_z Z$ intersects $\ker \omega_0 = \Sigma$. Since $\operatorname{rk} \Sigma = 1$, this implies that $TZ \supset T\Sigma$: with each $z \in Z$, the manifold Z contains the whole leaf of Σ passing through z. ■

Compatible Gauduchon metrics

DEFINITION: Let (M, Σ, ω_0) be a compact, complex manifold, and ω_0 a transversally Kähler form. A Gauduchon form ω is called **a Gauduchon form compatible with the transversally Kähler foliation,** if the projection to the leaf space of Σ (the leaf space is not always defined globally, but locally in M it always exists) is a Riemannian submersion.

REMARK: From now on, **we will always fix a Gauduchon form ω compatible with Σ, ω_0.**

CLAIM: At any given point of M, **there is a frame such that $\omega_0 = -\sqrt{-1}\, const \sum \theta_i \wedge \overline{\theta}_i$, and $\omega = -\sqrt{-1} \sum \theta_i \wedge \overline{\theta}_i - \sqrt{-1}\, \theta \wedge \overline{\theta}$, where $\theta, \theta_1, ..., \theta_n$ is an ω-orthonormal basis in $\Lambda^{1,0}(M)$, and all θ_i vanish on Σ.** ■

Leaf space of quasiregular foliations

REMARK: When Σ is quasiregular, M **is equipped with a holomorphic projection to the leaf space,** $\pi: M \longrightarrow X = M/\Sigma$. In this case **$\Sigma$ is a principal elliptic fibration**.

MAIN THEOREM: Let F be a stable coherent sheaf on a compact, complex manifold (M, Σ, ω_0), with a transversally Kähler, exact form, $\dim M > 2$. Assume that Σ is quasiregular, and let $\pi: M \longrightarrow X = M/\Sigma$ be the projection map. **Then $F = \pi^* F_0 \otimes L$, where F_0 is a coherent sheaf on M/Σ, and L a line bundle.**

Proof: Last slide.

COROLLARY: In these assumptions, **any coherent sheaf on M is filtrable**, that is, admits a filtration with rank 1 quotient sheaves.

REMARK: Filtrability is a very strong property! **It fails on almost all non-algebraic surfaces.**

Stability and transversally Kähler foliations (2)

THEOREM 1: Let (M, Σ, ω_0) be a compact, complex manifold, $\dim M > 2$, and ω_0 a transversally Kähler, exact form. Consider a vector bundle B with a Yang-Mills metric, $\deg_\omega B = 0$, and let ∇ denote the Yang-Mills connection. **Then the curvature Θ of ∇ satisfies $\Theta(x, \cdot) = 0$ for any $x \in \Sigma$.**

Proof: At a given point $m \in M$, let $\omega_0 = -\sqrt{-1} \sum \theta_i \wedge \overline{\theta}_i$, $\omega = -\sqrt{-1} \sum \theta_i \wedge \overline{\theta}_i - \sqrt{-1}\, \theta \wedge \overline{\theta}$, where $\theta, \theta_1, ..., \theta_n$ is an ω-orthonormal basis in $\Lambda^{1,0}(M)$ defined above. Write the curvature of B as

$$\Theta = \sum_{i \neq j} (\theta_i \wedge \overline{\theta}_j - \overline{\theta}_i \wedge \theta_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \overline{\theta}_i) \otimes a_i$$
$$+ \sum_i (\theta \wedge \overline{\theta}_i - \overline{\theta} \wedge \theta_i) \otimes b_i + \theta \wedge \overline{\theta} \otimes a,$$

with b_{ij}, b_i, a_i, $a \in \mathfrak{u}(B)$ being skew-Hermitian endomorphisms of B. Let $\Xi := \operatorname{Tr}(\Theta \wedge \Theta)$. Then $(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \operatorname{Tr}\left(-\sum b_i^2 + a\left(\sum a_i\right)\right)$. On the other hand, $\sum a_i + a = \Lambda\Theta = 0$, hence $(\sqrt{-1}\,)^n \Xi \wedge \omega_0^{n-2} = \operatorname{Tr}\left(-\sum b_i^2 - a^2\right)$. Since $\operatorname{Tr}(-a^2)$ is a positive definite form on $\mathfrak{u}(B)$, the integral $\int_M (\sqrt{-1}\,)^n \Xi \wedge \omega_0^{n-2}$ is non-negative, and positive unless b_i and a both vanish everywhere. If ω_0 is exact, this integral vanishes, and $\Theta(x) = 0$ for any $x \in \Sigma$. ■

Stable bundles on positive elliptic fibrations are equivariant

COROLLARY: In assumptions of Theorem 1, let $X \in \Sigma$ be a holomorphic vector field tangent to the foliation Σ. **Then ∇_X takes holomorphic sections of B to holomorphic sections.**

Proof: By definition, b is holomorphic if and only if $\nabla^{0,1}b = 0$. However,

$$\nabla_Y \nabla_X b = \nabla_{[Y,X]} b + \nabla_X \nabla_Y b + \Theta_{Y,X} b.$$

If $Y \in T^{0,1}M$, and b is holomorphic, we have $\nabla_Y b = 0$. Also, $[X,Y] \in T^{0,1}M$, because X is holomorphic, which also gives $\nabla_{[X,Y]} b = 0$. Finally, $\Theta_{X,Y} b = 0$ because $X \in \Sigma \subset \ker \Theta$. We have shown that $\nabla_Y \nabla_X b = 0$ for any $Y \in T^{0,1}M$.
■

COROLLARY: Let $M \xrightarrow{\pi} X$ be a positive principal elliptic fibration, E the elliptic curve, considered as a group, acting on the fibers of π, and $\tilde{E}$ its universal covering. **Then any ω-stable bundle on M is equipped with a natural $\tilde{E}$-equivariant structure.**

Proof: To define a $\tilde{E}$-equivariant structure, it would suffice to lift the vector fields tangent to E to holomorphic vector fields on $\operatorname{Tot} B$, compatible with the vector bundle structure. This was done in the previous theorem. ■

REMARK: To prove the Main Theorem, it remains to show that **$\tilde{E}$-equivariance implies that $B \otimes L = \pi^* B_0$,** for some line bundle L on M, and some bundle B_0 on $X = M/E$.

Line bundles on positive principal elliptic fibrations

Let E be an elliptic curve and $M \xrightarrow{\pi} X$ a positive principal E-fibration, $\dim M \geqslant 3$. Consider a line bundle L on M. Then $\tilde E$-equivariant structure on L defines a homomorphism $\pi_1(E) \longrightarrow \operatorname{Aut}(L) = \mathbb{C}^*$. We denote it χ_L. Since ∇ preserves the metric, χ_L takes values in $U(1)$, defining a map $\chi_L: \mathbb{Z}^2 \to U(1)$.

PROPOSITION: Let E be an elliptic curve and $M \xrightarrow{\pi} X$ a positive principal E-fibration, $\dim M \geqslant 3$. **Then for any character $\chi: \pi_1(E) \longrightarrow U(1)$. there exists a holomorphic line bundle L such that $L = \chi_L$.**

Proof. Step 1: Consider the commutative diagram with exact rows coming from the exponential exact sequence

$$\begin{array}{ccccc} H^1(M, \mathcal{O}_M) & \longrightarrow & \operatorname{Pic}_0(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ H^1(E, \mathcal{O}_E) & \longrightarrow & \operatorname{Pic}_0(E) & \longrightarrow & 0 \end{array}$$

The characters $\chi: \Gamma \longrightarrow U(1)$ are in bijective correspondence with the bundles from $\operatorname{Pic}_0(E)$, and the correspondence is provided by a unique flat connection on every $L \in \operatorname{Pic}_0(E)$. **It remains to show that the natural arrow $\operatorname{Pic}_0(M) \longrightarrow \operatorname{Pic}_0(E)$ is surjective.**

Step 2: From the diagram in Step 1, we obtain that surjectivity of the restriction $\operatorname{Pic}_0(M) \longrightarrow \operatorname{Pic}_0(E)$ **would follow if we prove that the restriction map $H^1(M, \mathcal{O}_M) \longrightarrow H^1(E, \mathcal{O}_E)$. is surjective.**

Step 3: Since $\operatorname{rk} H^1(E, \mathcal{O}_E) = 1$, surjectivity would follow if we prove that this map is non-trivial; equivalently, **if we show that the natural map** $\operatorname{Pic}_0(M) \to \operatorname{Pic}_0(E)$ **is non-zero.** Let L be a line bundle with curvature ω_0; then $c_1(L) = 0$, but this bundle is non-trivial, because were it trivial, we would have $\omega_0 = \partial\overline{\partial} f$, and

$$0 < \int_M \omega_0 \wedge \omega^{n-1} = \int_M \partial\overline{\partial} f \wedge \omega^{n-1} = \int_M f \wedge \partial\overline{\partial}\omega^{n-1} = 0.$$

Therefore, L is not a pullback of a line bundle on X, hence its restriction to the fibers of π is non-trivial. ■

Stable bundles on positive principal elliptic fibrations are pullbacks (up to a line bundle multiplier)

THEOREM: Let B be a ω-stable bundle on a compact, complex manifold equipped with a positive elliptic fibration $\pi: M \longrightarrow X$. **Then $B = \pi^* B_0 \otimes L$, where B_0 is a stable bundle on X, and L a holomorphic line bundle.**

Proof: Let $\rho: \tilde{E} \to \operatorname{Aut}(\operatorname{Tot} F)$ be the equivariant action constructed above, and Γ the kernel of the natural map $\tilde{E} \to E$. Since Γ acts on B by automorphisms, and the automorphisms of stable bundles are scalar, Γ acts on B as a character $\chi: \mathbb{Z}^2 \to U(1)$. However, by the previous proposition, any such character can be realized by a line bundle L. **Then $B \otimes L^{-1}$ is trivial on the leaves of π, which implies $B = \pi^* B_0 \otimes L$.** ■