Stable bundles on positive elliptic fibrations

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Estruturas geométricas em variedades,

IMPA, November 17, 2022

Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n-manifold is called **Gauduchon** if $dd^c\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique, up to a constant multiplier.

REMARK: If ω is Gauduchon, then (by Stokes' theorem) $\int_M \omega^{n-1} dd^c f = 0$ for any f. The curvature Θ_L of a holomorphic line bundle L is well-defined up to $dd^c \log |h|$, where h is a conformal factor. Therefore, **for any line bundle L, the number $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$ is well defined.**

REMARK: Unlike the Kähler case, $\deg_\omega L$ is a holomorphic invariant of L, and **not topological.**

DEFINITION: Given a torsion-free coferent sheaf F of rank r, let $\det F := \Lambda^r F^{**}$. From algebraic geometry it is known that $\det F$ is a line bundle. Define **the degree** $\deg_\omega F := \deg_\omega \det F = \int_M \operatorname{Tr} \Theta_F \wedge \omega^{n-1}$.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional Gauduchon manifold (M,ω), and $\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$. A torsion-free sheaf F is called **stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$, where Θ_B is its curvature.

THEOREM: (**Kobayashi-Hitchin correspondence**; Donaldson, Buchsdahl, Uhlenbeck-Yau, Li-Yau, Lübke-Teleman): Let B be a holomorphic vector bundle. **Then B admits a Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

REMARK: This result **was generalized to coherent sheaves** by Bando and Siu.

REMARK: **Stability is required** if you want to classify vector bundles or construct their moduli spaces.

Positivity for stable bundles

"Bogomolov's inequality": if $\deg B = 0$ and B is Yang-Mills, then $\operatorname{Tr}(\Theta_B \wedge \Theta_B) \wedge \omega^{n-2}$ is a positive volume form, vanishing only if the curvature Θ_B of B vanishes.

COROLLARY: A stable bundle B on a Kähler manifold M with $c_1(B) = 0, c_2(B) = 0$ is flat.

Today I will give a version of this statement on manifolds equipped with foliations, in particular, **when M is equipped with a positive elliptic fibration.**

Transversally Kähler foliations

DEFINITION: Semi-Hermitian form is a form $\omega \in \Lambda^{1,1}(M,\mathbb{R})$ such that $\omega(Ix,x) \geqslant 0$ for any $x \in TM$ **(the inequality is strict iff ω is Hermitian)**.

DEFINITION: A **foliation** on a complex manifold M is a complex sub-bundle $F \subset TM$, $\dim_R F = 2$, closed under commutator (usually it is assumed to be holomorphic). A foliation is called **transversally Kähler** if M is equipped with a closed semi-Hermitian form ω_0 such that $\omega_0(x,\cdot) = 0$ for any $x \in F$ and ω_0 is Hermitian on TM/F.

REMARK: On a compact Kähler n-manifold (M,ω), **a semi-Hermitian form ω_0 is never exact.** Indeed, $\int_M \omega_0 \wedge \omega^{n-1} > 0$, hence ω_0 cannot be exact. On compact, complex, non-Kähler manifolds, **transversally Kähler foliations with exact ω_0 are quite common.**

EXAMPLE: The classical Hopf surface is $H := \mathbb{C}^2\backslash 0/\mathbb{Z}$, where $\mathbb{Z}$ acts as a multiplication by a complex number λ, $|\lambda| > 1$. Clearly, H is diffeomorphic to $S^1 \times S^3$, and fibered over $\mathbb{C}P^1$ with fiber $\mathbb{C}^*/\langle\lambda\rangle$.

CLAIM: Let $\pi: H \longrightarrow \mathbb{C}P^1$ be the standard projection, and $\omega_0 := \pi^*\omega_{\mathbb{C}P^1}$ be a pullback of the Fubini-Study form. Clearly, **ω_0 is exact, because $H^2(H) = 0$** (by Künneth formula). Therefore, **H admits a transversally Kähler, exact form.**

Locally trivial elliptic fibrations.

DEFINITION: A principal elliptic fibration M is a complex manifold equipped with a free holomorphic action of a 1-dimensional compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

REMARK: It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

DEFINITION: Let $M \xrightarrow{\pi} X$ be a principal elliptic fibration, M compact. We say that M is **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact. **("Kähler class" is a cohomology class of a Kähler form).**

EXAMPLE: The classical Hopf surface introduced earlier.

EXAMPLE: A more general example is given by Tot$(L^*)/\langle\mathbb{Z}\rangle$, where L is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because $\pi^*(c_1(L)) = 0$, and $c_1(L)$ is a Kähler class.

Calabi-Eckmann manifolds

Calabi-Eckmann manifolds.
Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$.

Now, let $M := (\mathbb{C}^n\backslash 0)\otimes(\mathbb{C}^m\backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n\backslash 0)\otimes(\mathbb{C}^m\backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n\backslash 0) \otimes (\mathbb{C}^m\backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. Its total space M is called **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$.

REMARK: The map $M \longrightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ **is a principal elliptic fibration**.

REMARK: The pullback of a Kähler form from $\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ to M **is exact**, because $H^2(M) = 0$ (by Künneth formula).

Irregular and quasi-regular foliations

DEFINITION: A foliation is called **quasi-regular** if all its leaves are compact. If this is not so, it is called **irregular**. A foliation is called **regular** if all its leaves are compact, and the leaf space is smooth.

REMARK: The examples given above (Vaisman, Calabi-Eckmann) **are deformed naturally to irrregular foliated transversally Kähler manifolds.**

REMARK: Calabi-Eckmann manifolds were generalized by Lopez de Medrano, Verjovsky and Meersseman. The complex structure on Calabi-Eckmann can be deformed together with the foliation, giving **a transversally Kähler manifold with a foliation having non-compact leaves** ("LVM-manifolds").

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embedding denoted $\tau_i, \overline{\tau}_i$ and s real ones denoted σ_i, $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+}$ a free abelian subgroup $\mathcal{O}_K^{*,U}$ of rank s such that the quotient $\mathbb{R}^s/\mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to $\mathbb{R}^t$ as $\xi \longrightarrow \Big(\log(\sigma_1(\xi)), ..., \log(\sigma_t(\xi))\Big)$. Let $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: An Oeljeklaus-Toma manifold is a quotient $\mathbb{C}^t \times H^s/\Gamma$, where $\mathcal{O}_K^+$ acts on $\mathbb{C}^t \times H^t$ as

$$\zeta(x_1, ..., x_t, y_1, ..., y_s) = (x_1 + \tau_1(\zeta), ..., x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), ..., y_s + \sigma_s(\zeta)),$$

and $\mathcal{O}_K^{*,U}$ as $\xi(x_1, ..., x_t, y_1, ..., y_s) = (x_1, ..., x_t, \sigma_1(\xi)y_1, ..., \sigma_t(\xi)y_t)$

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times H^s/\Gamma$ **is a compact complex manifold,** without any non-constant meromorphic functions. When $t = 1$, it is locally conformally Kähler. When $s = 1, t = 1$, it is an Inoue surface of class S^0.

THEOREM: (Ornea-V.) Let M be an OT-manifold, $t = 1$. **Then M is equipped with a holomorphic 1-dimensional foliation and a transversally Kähler, exact form.**

Stability and transversally Kähler foliations

THEOREM: Let (M, Σ, ω_0) be a compact, complex manifold, $\dim M > 2$, and ω_0 a transversally Kähler, exact form. Let $\omega_1 := \omega_0 + \theta \wedge I(\theta)$, $\theta \in \Lambda^1(M)$ be a Hermitian form, and ω the corresponding Gauduchon form. Consider a vector bundle B with a Yang-Mills metric, $\deg_\omega B = 0$, and let ∇ denote the Yang-Mills connection. **Then the curvature Θ_B of ∇ satisfies $\Theta_B(x, \cdot) = 0$ for any $x \in \Sigma$.**

REMARK: The condition "Then the curvature Θ_B of ∇ satisfies $\Theta_B(x, \cdot) = 0$ for any $x \in \ker \omega_0$" means that (B, ∇) **is locally lifted from the leaf space of the foliation Σ.**

Leaf space of quasiregular foliations

REMARK: When Σ is quasiregular, **M is equipped with a holomorphic projection to the leaf space,** $\pi: M \longrightarrow X = M/\Sigma$. In this situation, the category of coherent sheaves can be described explicitly, in terms of a projective orbifold M/Σ.

THEOREM: Let F be a stable coherent sheaf on a be a compact, complex manifold (M, Σ, ω_0), with a transversally Kähler, exact form, $\dim M > 2$. Assume that Σ is quasiregular, and let $\pi: M \longrightarrow X = M/\Sigma$ be the projection map. **Then $F = \pi^* F_0 \otimes L$, where F_0 is a coherent sheaf on M/Σ, and L a line bundle.**

COROLLARY: In these assumptions, **any coherent sheaf on M is filtrable**, that is, admits a filtration with rank 1 quotient sheaves.

REMARK: Filtrability is a very strong property! **It fails on almost all non-algebraic surfaces.**

Stability and transversally Kähler foliations (2)

THEOREM: Let (M, Σ, ω_0) be a compact, complex manifold, $\dim M > 2$, and ω_0 a transversally Kähler, exact form. Let $\omega_1 := \omega_0 - \sqrt{-1}\theta\wedge\overline{\theta}$, $\theta \in \Lambda^{1,0}(M)$ be a Hermitian form, and ω the corresponding Gauduchon form. Consider a vector bundle B with a Yang-Mills metric, $\deg_\omega B = 0$, and let ∇ denote the Yang-Mills connection. **Then the curvature Θ_B of ∇ satisfies $\Theta_B(x, \cdot) = 0$ for any $x \in \Sigma$.**

Proof: At a given point $m \in M$, let $\omega_0 = -\sqrt{-1}\sum \theta_i \wedge \overline{\theta}_i$, $\omega = -\sqrt{-1}\sum \theta_i \wedge \overline{\theta}_i - \sqrt{-1}\,\theta \wedge \overline{\theta}$, where $\theta, \theta_1, ..., \theta_n$ is an orthonormal basis in $\Lambda^{1,0}(M)$. Write the curvature of B as

$$\Theta = \sum_{i\neq j} (\theta_i \wedge \overline{\theta}_j + \overline{\theta}_i \wedge \overline{\theta}_j) \otimes b_{ij} + \sum_i (\theta_i \wedge \overline{\theta}_i) \otimes a_i$$
$$+ \sum_i (\theta \wedge \overline{\theta}_i + \overline{\theta} \wedge \overline{\theta}_i) \otimes b_i + \theta \wedge \overline{\theta} \otimes a,$$

with b_{ij}, b_i, a_i, $a \in \mathfrak{u}(B)$ being skew-Hermitian endomorphisms of B. Let $\Xi := \operatorname{Tr}(\Theta \wedge \Theta)$. Then $(\sqrt{-1})^n \Xi \wedge \omega_0^{n-2} = \operatorname{Tr}\left(-\sum b_i^2 + a\,(\sum a_i)\right)$. On the other hand, $\sum a_i + a = \Lambda\Theta = 0$, hence $(\sqrt{-1}\,)^n \Xi \wedge \omega_0^{n-2} = \operatorname{Tr}\left(-\sum b_i^2 - a^2\right)$. Since $\operatorname{Tr}(-a^2)$ is a positive definite form on $\mathfrak{u}(B)$, the integral $\int_M (\sqrt{-1}\,)^n \Xi \wedge \omega_0^{n-2}$ is non-negative, and positive unless b_i and a both vanish everywhere. If ω_0 is exact, this integral vanishes, and $\Theta(x) = 0$ for any $x \in \Sigma$. ■

Foliation by group orbits

REMARK: The condition $\deg_\omega B = 0$ **can be rectified by a tensor multiplication with an appropriate line bundle of non-zero degree.**

THEOREM: Let (M, Σ, ω_0) be a compact, complex manifold, $\dim M > 2$, and ω_0 a transversally Kähler, exact form. Let ρ denote an action of $\mathbb{C}$ on M. Assume that its orbits are leaves of Σ, and that the form ω_0 is ρ-invariant. **Then any stable bundle (or coherent sheaf) B on M is ρ-equivariant.**

Proof: When $\deg_\omega B = 0$, the bundle B is flat on the leaves of Σ, and the parallel transport along leaves is compatible with the connection. When $\deg_\omega B \neq 0$, we multiply B by a ρ-equivariant line bundle L of degree $\frac{-\deg B}{\operatorname{rk} B}$. For L one could take a topologically trivial line bundle with curvature $const\omega_0$.

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