Toric Fibrations on homogeneous complex manifolds

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HSE, Geometric Structures on Manifolds

Homogeneous complex manifolds

Definition 1.1: A complex manifold M is called **homogeneous** if its automorphism group acts transitively.

Examples of compact homogeneous manifolds:

0. Flag spaces and partial flag spaces.

1. Calabi-Eckmann and Hopf manifolds.

2. Tori.

3. Let G be a compact, even-dimensional Lie group. Then G **admits a left-invariant complex structure** (H. Samelson, 1953).

Remark 1.2: Compact homogeneous complex manifolds are usually non-Kähler (exception: partial flag spaces, tori, their products and finite quotients).

Hopf surface

The (classical) Hopf surface. Fix $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Consider the quotient $H = (\mathbb{C}^2 \backslash 0)/\langle \mathbb{Z} \rangle$, with $\mathbb{Z}$ acting on $\mathbb{C}^2$ by $(x, y) \longrightarrow (\alpha x, \alpha y)$. It is called **the Hopf surface**. Topologically the Hopf surface is isomorphic to $S^1 \times S^3$ (hence, non-Kähler). The elliptic curve $T^2 = \mathbb{C}^*/\langle \alpha \rangle$ acts on H by $t, (x, y) \longrightarrow (tx, ty)$. This action is free, and its quotient is $\mathbb{C}P^1$. The Hopf surface is a **principal elliptic fibration.** Topologically, it's a product of a Hopf fibration $S^3 \longrightarrow S^2$ and a circle.

Calabi-Eckmann manifolds

Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$.

Now, let $M := (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. Then M is called **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$. The group $U(n) \times U(m)$ acts on M transitively.

We obtained a homogeneous complex structure on $S^{2n-1} \times S^{2m-1}$.

It is non-Kähler, because $H^2(M) = 0$.

Definition:
A complex principal toric fibration (bundle) M is a complex manifold equipped with a free holomorphic action of a compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

Can consider this notion in smooth category as well (remove "complex" and "holomorphic" from this definition).

To trivialize a principal group bundle it means to find a section (holomorphic section for complex trivialization, smooth for topological).

Topology of principal toric bundles

(not necessarily complex)

A principal T^n-bundle over X is defined topologically by $H^1(X, \mathbb{T})$, where $\mathbb{T}^n$ is a sheaf of smooth T^n-valued functions on X. An exact sequence

$$0 \longrightarrow \Gamma \longrightarrow C^\infty(M)^n \longrightarrow \mathbb{T}^n \longrightarrow 0,$$

gives $H^1(\mathbb{T}^n) = H^2(M, \Gamma)$, where $\Gamma = \pi_1(T)$ If Denote by

$$\tau: \ H^1(T, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{Z})$$

the map which corresponds to the $H^1(X, \mathbb{T}^n)$-class of a fiber bundle.

A principal fiber bundle is determined, up to a topological equivalence, by this invariant. Also, any such τ corresponds to a principal fiber bundle.

Example: A principal S^1-bundle is a determined by its Chern class c_1 in $H^2(M)$.

The transfer map

Using the Leray-Serre spectral sequence, it is easy to express the cohomology of M in terms of $H^*(X)$ and the Chern classes. This gives an exact sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(M) \longrightarrow H^1(T) \xrightarrow{d_2} H^2(X) \longrightarrow H^2(M)$$

with d_2 (a differential in Leray-Serre spectral sequence), called **the transfer map**. It is easy to see that $\tau = d_2$.

Examples of principal toric bundles (in smooth category):

1. "Hopf fibration". S^3 fibered over S^2, with fiber S^1, and the Chern class 1. It is a total space of $U(1)$-bundle over $\mathbb{C}P^1$, which is denoted as $\mathcal{O}(-1)$.

2. A generalization of this example. S^{2n+1} is fibered over $\mathbb{C}P^n$, with fiber S^1. Again, it is a total space of $U(1)$-bundle, corresponding to $\mathcal{O}(-1)$.

3. G a Lie group, $T \subset G$ a torus, G fibered over G/T.

4. Nilmanifolds (manifolds with transitive action of a nilpotent Lie group) always admit principal toric fibrations.

Complex principal toric bundles (complex manifolds with a free, holomorphic action of a complex torus T).

1. **The (classical) Hopf surface**. Fix $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Consider the quotient $H = (\mathbb{C}^2 \backslash 0)/\langle\mathbb{Z}\rangle$, with $\mathbb{Z}$ acting on $\mathbb{C}^2$ by $(x,y) \longrightarrow (\alpha x, \alpha y)$. It is called **the Hopf surface**. Topologically the Hopf surface is isomorphic to $S^1 \times S^3$ (hence, non-Kähler). The elliptic curve $T^2 = \mathbb{C}^*/\langle\alpha\rangle$ acts on H by $t, (x,y) \longrightarrow (tx, ty)$. This action is free, and its quotient is $\mathbb{C}P^1$. The Hopf surface is a principal elliptic fibration. Topologically, it's a product of a Hopf fibration $S^3 \longrightarrow S^2$ and a circle.

2. **A generalization of this example.** Let X be a complex manifold, and L a holomorphic line bundle on X. Consider a principal $\mathbb{C}^*$-bundle Tot(L^*) over X (total space of L without a zero section). Taking a quotient

$$\mathrm{Tot}(L^*)/\langle\mathbb{Z}\rangle, \quad \text{with } \mathbb{Z} \text{ acting as } v \mapsto \alpha v,$$

we obtain, again, a principal elliptic bundle, with fiber $T^2 = \mathbb{C}^*/\langle\alpha\rangle$. When $X = \mathbb{C}P^1$, $L = \mathcal{O}(-1)$, this gives a Hopf surface.

Complex principal toric bundles (cont.)

Using the Leray-Serre spectral sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(M) \longrightarrow H^1(T) \xrightarrow{d_2} H^2(X)$$

and the fact that $\operatorname{im} d_2 = \langle c_1(L) \rangle$, we obtain that $H^1(M)$ is odd-dimensional (hence, cannot be Kähler), for any $M = \operatorname{Tot}(L^*)/\langle \mathbb{Z} \rangle$, with $c_1(L)$ nonzero over $\mathbb{Q}$.

3. **Calabi-Eckmann manifolds.**
Fix $\alpha \in \mathbb{C}$, α non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle \alpha \rangle$.

Now, let

$$M = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G,$$

with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. The fibration $M \longrightarrow \mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1}$ is called **the Calabi-Eckmann fibration**, its total space M **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$.

We obtained a homogeneous complex structure on $S^{2n-1} \times S^{2m-1}$.

It is non-Kähler, because $H^2(M) = 0$.

Borel-Remmert-Tits theorem:
Let M be a compact, complex, simply connected homogeneous manifold ("homogeneous" means that $G = \operatorname{Aut}(M)$ acts on M transitively). Then M is a principal toric fibration, with a base which is a homogeneous, rational projective manifold.

Proof: Let $K^{-1} = \Lambda^{top}(TM)$ be the anticanonical class of M. Since TM is globally generated, the same is true for K^{-1}. This gives a G-invariant morphism

$$M \xrightarrow{\pi} \mathbb{P}H^0(K^{-1}).$$

The fibers F of π are homogeneous with trivial canonical class, and its base is homogeneous and projective (hence, rational). The fundamental group of F is a quotient of $\pi_2(X)$, as follows from the long exact sequence of homotopy groups for a Serre's fibration:

$$\pi_2(X) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) = 0$$

Therefore, $\pi_1(F)$ is abelian. **It remains to show that it is a torus.**

Lemma: Let F be a compact, complex, homogeneous manifold with $\pi_1(F)$ abelian and a trivial anticanonical class K^{-1}. Then F is a torus.

Proof: The sheaf of holomorphic vector fields on M is globally generated. Taking a vector field v_1 and multiplying it by general vector fields $v_2, ...v_n$, we obtain a section of K^{-1}, which is non-zero for general v_i, and therefore non-degenerate. We obtain that v_i are linearly independent everywhere. Taking the corresponding flows of diffeomorphisms, we obtain that F is a quotient of a holomorphic Lie group G by a cocompact lattice. Since $\pi_1(F)$ is abelian, G is commutative, and T is a torus.

Positive line bundles.

Let X be a complex manifold, and L a holomorphic line bundle. L is called **positive**, or **ample** if for sufficiently big N, $L^{\otimes N}$ is globally generated, and, moreover, the natural map

$$X \longrightarrow \mathbb{P}(H^0(L^{\otimes N}))$$

is an embedding. In this case $L^{\otimes N}$ is called **very ample**.

Theorem (Kodaira-Nakano):
A holomorphic line bundle is ample if and only if it admits a Hermitian metric, with curvature Θ which satisfies $-\sqrt{-1}\,\Theta(z,\overline{z}) > 0$ for any non-zero vector $z \in T^{1,0}(M)$.

This means that $-\sqrt{-1}\,\Theta(\cdot, I\cdot)$ is a Kähler metric on X.

Positive elliptic fibrations.

Definition: Let $M \xrightarrow{\pi} X$ be an elliptic fibration, M compact. We say that M is **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact. "Kähler class" is a cohomology class of a Kähler form.

Examples:

1. Hopf manifold, $H^2(M) = 0$, hence positive

2. Calabi-Eckmann manifold (same)

3. $SU(3)$ is elliptically fibered over the flag manifold $F(2,3)$, also $H^2(M) = 0$.

4. $\operatorname{Tot}(L^*)/\langle\mathbb{Z}\rangle$, where L is an ample line bundle. Such manifold is called **a regular Vaisman manifold**. It is positive, because $\pi^*(c_1(L)) = 0$, and $c_1(L)$ is a Kähler class.

It is possible to interpret τ as a "curvature class" of a fibration, and when it is Kähler, we can say that a fibration is positive. This happens precisely when the image of τ contains a Kähler class.

Subvarieties of positive elliptic fibrations

Theorem: Let $M \xrightarrow{\pi} X$ be a positive elliptic T fibration, and $Z \subset M$ be a subvariety, of positive dimension m. Then Z is T-invariant.

Proof: Let $\omega_0 = \pi^* \omega$ be a pullback of a Kähler form which is exact. Then

$$\int_Z \omega_0^m = 0.$$

On the other hand, all eigenvalues of $\omega_0\Big|_Z$ are non-negative, and all are positive, unless Z is tangent to the action of T. In a point where Z is not tangent to T, the form ω_0^m is positive, and in this case the integral $\int_Z \omega_0^m$ is also positive.

A similar result is true for stable coherent sheaves.

Theorem: Let $M \xrightarrow{\pi} X$, $\dim_{\mathbb{C}} X > 1$, be a positive elliptic T fibration, and F a stable reflexive sheaf on M. Then $F \cong L \otimes \pi^* F_0$, where L is a line bundle, and F_0 a stable coherent sheaf on X.

Positive toric fibrations

Definition: Let $M \xrightarrow{\pi} X$ be a complex principal toric fibration, M compact, with fiber T. Assume that the image of $\tau: H^1(T,\mathbb{C}) \longrightarrow H^2(X,\mathbb{C})$ contains a Kähler form. Then the fibration $M \xrightarrow{\pi} X$ is called **convex**. *NB:* Can define convexity for arbitrary fiber bundles.

Consider a holomorphic quotient $T_1 = T/T_2$ of T. Taking the quotient space M/T_2, we obtain a complex principal toric fibration, with fiber T_1.

Assume that for all $T_1 = T/T_2$, $\dim T_1 > 0$, the induced fibration $M/T_2 \longrightarrow X$ is also convex. Then $M \xrightarrow{\pi} X$ is called **positive**.

Example. Let M be a complex, compact homogeneous manifold with $H^2(M) = 0$ (e.g. a Lie group), and $M \xrightarrow{\pi} X$ the Borel-Remmert-Tits toric fibration. Assume that the fibers of π have no proper subtori (easy to insure by taking a generic invariant complex structure). Then M is positive.

Theorem. Consider an irreducible complex subvariety $Z \subset M$ of a positive principal toric fibration $M \xrightarrow{\pi} X$, with fiber T. Then Z is T-invariant, or is contained in a fiber of π.

Proof: 1. For any positive-dimensional subvariety $Z_0 \subset X$, the restriction of π to Z_0 has no multisections (because $\int_Z \omega_0^m$ must vanish).

2. Given a space A (of Fujiki class C) with an action of T, consider an associated fiber bundle $M \times_T A$ over X. Unless T acts on A trivially, $M \times_T A$ is also convex, hence admits no multisections.

3. If $Z \subset M$ is not T-invariant, it provides us with a multisection from X to $M \times_T A$, where A is the space of deformations of the fiber $Z \cap \pi^{-1}(t_0)$. It is of Fujiki class C, hence convex. Cannot have multisections! Contradiction.