Principal toric fibrations on homogeneous complex manifolds

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Homogeneous complex manifolds

DEFINITION: A complex manifold M is called **homogeneous** if its automorphism group acts transitively.

Examples of compact homogeneous manifolds:

0. Flag spaces and partial flag spaces.

1. Calabi-Eckmann and Hopf manifolds.

2. Tori.

3. Let G be a compact, even-dimensional Lie group. Then G **admits a left-invariant complex structure** (H. Samelson, 1953).

REMARK: Compact homogeneous complex manifolds are usually non-Kähler (exception: partial flag spaces, tori, their products and finite quotients).

Hopf surface

The (classical) Hopf surface. Fix $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Consider the quotient $H = (\mathbb{C}^2 \backslash 0)/\langle\mathbb{Z}\rangle$, with $\mathbb{Z}$ acting on $\mathbb{C}^2$ by $(x, y) \longrightarrow (\alpha x, \alpha y)$. It is called **the Hopf surface**. Topologically the Hopf surface is isomorphic to $S^1 \times S^3$ (hence, non-Kähler). The elliptic curve $T^2 = \mathbb{C}^*/\langle\alpha\rangle$ acts on H by $t, (x, y) \longrightarrow (tx, ty)$. This action is free, and its quotient is $\mathbb{C}P^1$. The Hopf surface is a **principal elliptic fibration.** Topologically, it's a product of a Hopf fibration $S^3 \longrightarrow S^2$ and a circle.

The Calabi-Eckmann manifold is a complex manifold diffeomorphic to $S^{2n-1} \times S^{2m-1}$ with transitive holomorphic action of $U(n) \times U(m)$. It is defined as follows.

Calabi-Eckmann manifolds

Fix $\alpha \in \mathbb{C}$, $\operatorname{Im}\alpha > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. Clearly, for each $(x,y) \in \mathbb{C}^* \times \mathbb{C}^*$ there exists $t \in \mathbb{C}$ such that $e^t x = 1$. This defines t up to $2\pi\sqrt{-1}$, hence y is defined up a multiplier λ^n, where $\lambda = e^{\sqrt{-1}\,2\pi\alpha}$. Since $|\lambda| > 1$, the quotient $\frac{\mathbb{C}^* \times \mathbb{C}^*}{e^t \times e^{\alpha t}} = \frac{\mathbb{C}^*}{\langle\lambda\rangle}$ is an elliptic curve

Now, let $M := (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x,y) \longrightarrow (t_1 x, t_2 y)$. Clearly, M is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. This complex manifold is called **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$. The group $U(n) \times U(m)$ acts on M transitively.

It is non-Kähler, because $H^2(M) = 0$.

Principal toric fibrations

DEFINITION: A complex principal toric fibration M is a complex manifold equipped with a free holomorphic action of a compact complex torus T.

Such a manifold is fibered over M/T, with fiber T.

It is a principal T-bundle: all fibers are identified with T, with T acting on fibers freely.

To trivialize a principal group bundle it means to find a section.

Borel-Remmert-Tits theorem

Borel-Remmert-Tits theorem: Let M be a compact, complex, simply connected homogeneous manifold. Then M is a principal toric fibration, with a base which is a homogeneous, rational projective manifold.

QUESTION: Let $K^{-1} = \Lambda_{\mathbb{C}}^{\dim M}(TM)$ be the anticanonical class of M. Since TM is globally generated, the same is true for K^{-1}. This gives a G-invariant morphism

$$M \xrightarrow{\pi} \mathbb{P}H^0(K^{-1}).$$

The fibers F of π are homogeneous with trivial canonical class, and its base is homogeneous and projective (hence, rational). The fundamental group of F is a quotient of $\pi_2(X)$, as follows from the long exact sequence of homotopy groups for a Serre's fibration:

$$\pi_2(X) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) = 0$$

Therefore, $\pi_1(F)$ is abelian. **It remains to show that it is a torus.**

Homogeneous manifolds with trivial canonical class

LEMMA: Let F be a compact, complex, homogeneous manifold with $\pi_1(F)$ abelian and a trivial anticanonical class K^{-1}. **Then F is a torus.**

Proof: The sheaf of holomorphic vector fields on M is globally generated. Taking a vector field v_1 and multiplying it by general vector fields $v_2, ...v_n$, we obtain a section of K^{-1}, which is non-zero for general v_i, and therefore non-degenerate. We obtain that v_i are linearly independent everywhere. Taking the corresponding flows of diffeomorphisms, we obtain that F is a quotient of a holomorphic Lie group G by a cocompact lattice. **Since $\pi_1(F)$ is abelian, G is commutative, and T is a torus.** ■

Positive elliptic fibrations

DEFINITION: Let $M \stackrel{\pi}{\longrightarrow} X$ be an elliptic fibration, M compact. We say that M is a **positive elliptic fibration**, if for some Kähler class ω on X, $\pi^*\omega$ is exact. ("Kähler class" is a cohomology class of a Kähler form.)

Examples:

1. **Hopf manifold**, $H^2(M) = 0$, hence positive.
2. **Calabi-Eckmann manifold** (same).
3. $SU(3)$ is elliptically fibered over the flag manifold $F(2,3)$, also $H^2(M) = 0$.

Subvarieties of positive elliptic fibrations

Theorem: Let $M \stackrel{\pi}{\longrightarrow} X$ be a positive elliptic T fibration, and $Z \subset M$ a subvariety of positive dimension m. **Then Z is T-invariant.**

Proof: Let $\omega_0 = \pi^*\omega$ be a pullback of a Kähler form which is exact. Then

$$\int_Z \omega_0^m = 0.$$

On the other hand, all eigenvalues of $\omega_0\Big|_Z$ are non-negative, and positive if Z is transversal to the action of T. Since $\int_Z \omega_0^m = 0$, the variety Z is tangent to T everywhere. ■

Positive toric fibrations

DEFINITION: Let $M \xrightarrow{\pi} X$ be a complex principal toric fibration, M compact, with fiber T. We say that π is **convex** if $\pi^*\omega$ is exact for some Kähler form ω. We say that π is **positive** if for any proper complex subtorus $T' \subset T$, the corresponding quotient fibration $M/T' \longrightarrow X$ is convex.

EXAMPLE: Let M be a complex, compact homogeneous manifold with $H^2(M) = 0$ (e.g. a Lie group), and $M \xrightarrow{\pi} X$ the Borel-Remmert-Tits toric fibration. Assume that the fiber of π have no proper subtori (easy to insure by taking a generic invariant complex structure). **Then M is positive.**

Subvarieties in principal toric fibrations

THEOREM: Consider an irreducible complex subvariety $Z \subset M$ of a positive principal toric fibration $M \stackrel{\pi}{\longrightarrow} X$, with fiber T. **Then Z is T-invariant, or is contained in a fiber of π.**

Proof: 1. For any positive-dimensional subvariety $Z_0 \subset X$, the restriction of π to Z_0 **has no multisections** (because $\int_Z \omega_0^m$ must vanish).

2. Given a Kähler manifold A with an action of T, consider an associated fiber bundle $M \times_T A$ over X. Unless T acts on A trivially, the fibration $M \times_T A \mapsto X$ is also convex, hence **it admits no multisections.**

3. If $Z \subset M$ is not T-invariant, it provides us with a multisection from X to $M \times_T A$, where A is the Barlet space of deformations of the variety $Z \cap \pi^{-1}(t_0)$ in the torus. It is convex (step 2). **Then it cannot have multisections!** Contradiction. ■

Open questions

THEOREM: Let $M \xrightarrow{\pi} X$, $\dim_{\mathbb{C}} X > 1$, be a positive elliptic T fibration, and F a stable reflexive sheaf on M. **Then $F \cong L \otimes \pi^* F_0$, where L is a line bundle, and F_0 a stable coherent sheaf on X.**

COROLLARY: Every coherent sheaf on M **is an extension of sheaves obtained as $L \otimes \pi^* F_0$.**

DEFINITION: A coherent sheaf is called **filtrable** if it is an extension of coherent sheaves of rank 1 and 0.

COROLLARY: A coherent sheaf on a total space of a principal elliptic fibration **is filtrable.**

QUESTION: Is there a similar result for positive torus fibrations?