Primitive line bundles and multiple fibers on holomorphic Lagrangian fibrations

Misha Verbitsky

Estruturas geométricas em variedades,
IMPA, July 18, 2024

(joint work with Ljudmila Kamenova)

Plan.

1. Enoki-Mourugane-Takegoshi-Demailly-Peternell-Schneider vanishing theorem.

2. Hirzebruch-Riemann-Roch for hyperkähler manifolds.

3. Primitivity of pullbacks of hyperplane sections in Lagrangian fibrations

4. Multiple fibers and multiplicities

Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}:\ V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0}b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0}b, b) = \operatorname{Re} \eta|b|^2$. **This gives $\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

COROLLARY: If $g' = e^{2f}g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

Enoki vanishing theorem

DEFINITION: A real $(1,1)$-form η on a complex manifold M is called **positive** if $\eta(x, Ix) \geqslant 0$ for all real tangent vectors x.

REMARK: This is a case of so-called **"French positivity"**: in this terminology, 0 is a positive form.

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we do not use) is due to Demailly, Peternell and Schneider.

THEOREM: Let (M, I, ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, K its canonical bundle, and L a holomorphic line bundle on M equipped with a Hermitian metric h. Assume that the curvature Θ of L is a positive form on M. Then **the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map**

$$H^0(\Omega^{n-i} M \otimes L) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L).$$

Here ω is considered as an element in $H^1(\Omega^1 M)$, and multiplication by ω maps $H^k(\Omega^{n-l} M \otimes L)$ to $H^{k+1}(\Omega^{n-l+1} M \otimes L)$.

Enoki vanishing theorem (2)

The proof of Enoki vanishing is implied by Serre's duality and the following lemma.

LEMMA: Let (M,I,ω) be a compact Kähler manifold, and L holomorphic Hermitian line bundle with $-\Theta_L$ positive, where Θ_L is its curvature. Let $\partial: \Lambda^{a,b}(M)\otimes L \longrightarrow \Lambda^{a+1,b}(M)\otimes L$ denote the $(1,0)$-part of the Chern connection map $d_\nabla: \Lambda^p(M)\otimes L \longrightarrow \Lambda^{p+1}(M)\otimes L$, and $\Delta_\partial, \Delta_{\overline{\partial}}$ the corresponding Laplacians. Consider a $\Delta_{\overline{\partial}}$-harmonic $(0,k)$-form $\eta \in \Lambda^{0,k}(M)\otimes L$. **Then $\partial\eta = 0$. Moreover, $\eta\wedge\Theta_L = 0$.**

Sketch of a proof. Step 1: The proof is not much different from the standard proof of Kodaira-Nakano vanishing. The difference between the Laplacians is

$$\Delta_{\overline{\partial}} - \Delta_\partial = -[L_\Theta, \Lambda],$$

where L_Θ is an operator of a multiplication by Θ, and Λ Hermitian adjoint to the operator of multiplication by the Kähler form.

Enoki vanishing theorem (3)

Step 2: For any $(0,k)$-form $e = \overline{\xi}_{i_1} \wedge ... \wedge \overline{\xi}_{i_k}$, we have $[L_\Theta, \Lambda]e = \sum \alpha_{j_p} e$, where the sum is taken over all α_{j_p} with $j_p \notin \{i_1, i_2, ..., i_k\}$. Therefore, $\Delta_{\overline{\partial}} = \Delta_\partial + A$, where the operator $A(e) = \sum \alpha_{j_p} e$ is positive and self-adjoint on $\Lambda^{0,k}(M) \otimes L$.

Step 3: This gives, for any $\eta \in \ker \Delta_{\overline{\partial}}$, that $\Delta_\partial(\eta) = 0$ and $A(\eta) = 0$. In other words, for any $\eta \in \ker \Delta_{\overline{\partial}}$, the form $\overline{\eta}$ is holomorphic, and η is a sum of monomials $e = \overline{\xi}_{i_1} \wedge ... \wedge \overline{\xi}_{i_k}$ containing all $\overline{\xi}_{i_p}$ with $\alpha_{i_p} \neq 0$. This is equivalent to $e \wedge \Theta = 0$. ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

Hyperkähler manifolds of maximal holonomy

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

Further on, **all hyperkähler manifolds are assumed to be of maximal holonomy**.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ -\frac{n-1}{2n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q has signature $(3, b_2 - 3)$. It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Hirzebruch-Riemann-Roch formula

DEFINITION: Let B be a holomorphic vector bundle (or a coherent sheaf). The **holomorphic Euler characteristic** is $\chi(L) := \sum_i (-1)^i H^i(M,B)$.

THEOREM: (Riemann-Roch-Hirzebruch) Let M be a compact complex manifold, and B a holomorphic vector bundle. **Theb $\chi(B)$ can be expressed through the the Chern classes of TM and B,** $\chi(M) = \int_M td(TM) \wedge ch(B)$ where td is the the Todd polynomial on Chern classes of TM,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

and $ch(B)$ its Chern character,

$$ch(B) = 1 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + ...$$

■

Hirzebruch-Riemann-Roch formula and BBF form

THEOREM: (Huybrechts) Let M be a hyperkähler manifold, $\dim_C M = 2n$ and L a holomorphic line bundle. **Then $\chi(L) = \sum a_i q(c_1(L))^i$, where the coefficients a_i are constants depending on the topology of M.**

Proof. Step 1: Let A^* be the subalgebra in cohomology generated by $H^2(M)$. **Then $A^{2i} \cong \operatorname{Sym}^i(H^2(M))$ up to the middle degree, and $A^{n+i} \cong \operatorname{Sym}^{n-i}(H^2(M))$; there is an $O(H^2(M))$-action on cohomology, and the multiplication is $O(H^2(M))$-invariant (V., 1995).**

Step 2: All Chern classes of TM are $O(H^2(M))$-invariant, but there is only one (up to a constant multiplier) $O(H^2(M))$-invariant functional on $\operatorname{Sym}^{2i}(H^2(M))$. On the class $\eta^{2i} \in H^{4i}(M)$ this functional takes value $q(\eta, \eta)^i$. Therefore, **all L-dependent coefficients in the Hirzebruch-Riemann-Roch formula for $\chi(L)$ are expressed through $q(c_1(L))$.** ■

COROLLARY: Let L be a line bundle on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$. Assume that $q(c_1(L)) = 0$. **Then $\chi(L) = n + 1$.**

Proof: Indeed, $\chi(L) = \chi(\mathcal{O}_M) = n + 1$, with the second equality implied by Bochner's vanishing theorem. ■

Second cohomology of a hyperkähler manifold is torsion-free

CLAIM: Let M be a hyperkähler manifold of maximal holonomy. **Then $H^2(M)$ is torsion-free.**

Proof: The universal coefficients formula gives the exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $H_1(X,\mathbb{Z}) = 0$ for a maximal holonomy hyperkähler manifold, **this gives an isomorphism $H^2(X;\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z})$, hence the torsion vanishes.** ■

Lagrangian fibrations

THEOREM: (Matsushita) Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a surjective holomorphic map, with $0 < \dim X < \dim M$. **Then π is a Lagrangian fibration** (that is, has holomorphic Lagrangian fibers).

THEOREM: (Hwang) In these assumptions, **X is biholomorphic to $\mathbb{C}P^n$ when it is smooth.**

CONJECTURE: X is biholomorphic to $\mathbb{C}P^n$ **when it is normal.**

THEOREM: (Matsushita)
Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a Lagrangian fibration, with X normal. **Then $H^*(X, \mathbb{Q}) \cong H^*(\mathbb{C}P^n, \mathbb{Q})$.**

REMARK: General fibers of π are Abelian varieties (projective complex tori), by Arnold-Liouville. Conversely, as shown by Hwang-Weiss, **any Lagrangian complex torus in M is a fiber of a Lagrangian fibration.**

Primitivity and vanishing of cohomology

Lemma 2: Let B be a vector bundle equipped with a filtration $0 = B_0 \subset B_1 \subset ... \subset B_k = B$. Assume that $H^0(B_i/B_{i-1}) = 0$. **Then $H^0(B) = 0$.**

Proof: Do this as an exercise. ■

THEOREM: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L is trivial on all smooth fibers of π.**

Proof. Step 1: Let F be a smooth fiber of π, which is an abelian variety by Arnol'd-Liouville. Then $T^*M|_F$ is an extension of a trivial bundle TF with another trivial bundle $NF = T^*F$. **For any non-trivial line bundle $L \in \operatorname{Pic}_0(F)$, we have $H^0(L \otimes TF) = 0$ and $H^0(L \otimes NF) = 0$, which implies that $H^0(L \otimes T^*M|_F) = 0$.** Similarly one $H^0(L \otimes \Lambda^k T^*M|_F) = 0$ (Lemma 2).

Step 2: Unless L is trivial on F, we have $H^0(L \otimes \Lambda^k T^*M|_F) = 0$, which implies that $H^0(L \otimes \Lambda^* M) = 0$. By Enoki (also Mourugane-Takegoshi-Demailly-Peternell-Schneider) theorem, **this implies that $H^i(L) = 0$, hence $\chi(L) = 0$, contradicting the formula $\chi(L) = n + 1$ (Huybrechts).** ■

Fiberwise monodromy of a line bundle

Proposition 1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L admits a connection ∇ which is flat on each restriction $L|_F$ to the fiber of π.**

Proof: Choose a constant metric h^k on $L^{\otimes k}|_F = \mathcal{O}_F$ and let h be its k-th root, which is a metric on $L|_F$. **Since h^k is constant, its curvature is flat, and the Chern connection ∇ associated with h is also flat.**

DEFINITION: Fiberwise monodromy of L is its monodromy on the fibers of π.

Remark 1: Clearly, **$L = \pi^* L_0$ if the fiberwise monodromy of L on each fiber is trivial.**

Clemens-Persson retraction result

THEOREM: (Clemens-Persson)
Let $\pi: M \to X$ be a proper holomorphic map, and F_x its fiber in $x \in X$. **Then there exists a neighbourhood $U \subset X$ of x and a continuous retraction of $\pi^{-1}(U)$ to F_x.** ■

THEOREM: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. Given a special fiber F_x, consider the Clemens-Persson retraction map $\Phi: \pi^{-1}(U) \longrightarrow F_x$. Let F be a smooth fiber of π. Assume that $\Phi: F \longrightarrow F_x$ is surjective on fundamental groups. **Then L is trivial on F_x.**

Proof: Let $\gamma_x \in \pi_1(F_x)$ be a loop, and $\gamma \in \pi_1(F)$ be a loop such that $\Phi(\gamma) = \gamma_x$. Consider the homotopy Φ_t contracting Id to Φ, with $\Phi_0 = \mathrm{Id}$ and $\Phi_1 = \Phi$, and denote by μ_t the monodromy of the connection ∇ (Proposition 1) along $\Phi_t(\gamma)$. By construction, μ_t is continuous in t, and trivial for all $t \neq 1$, hence **the monodromy of ∇ along γ is trivial, and L is trivial on fibers of π by Remark 1.** ■

Surjectivity in codimension 1

DEFINITION: Recall that a line bundle H is called **primitive** if H is not isomorphic to a non-trivial tensor power of another line bundle.
Theorem 1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to X$, and H a primitive line bundle on X. Let $D \subset X$ be **the discriminant** of π, that is, the set of singular values of π. Assume that for a general $x \in D$, the Clemens-Persson map $\Phi: F \longrightarrow F_x$ is surjective on fundamental groups. **Then $\pi^* H$ is primitive.**

Proof: Consider a primitive line bundle L on M such that $\pi^* H = L^{\otimes k}$. Let $D_0 \subset D$ be the set of all $x \in D$ such that $L|_{F_x}$ is trivial. This set is Zariski open, hence **its complement $D_1 \subset D$ has codimension 1 in D and codimension 2 in X.**

Step 2: Let $Z := \pi^{-1}(D_1)$, and $j: M \backslash Z \longrightarrow M$ the open embedding. **Then $L = j_* j^* L$ by "Serre's condition S2" (OSS, Ch. II, Lemma 1.1.12)**. This almost immediately implies that L is a pullback of a bundle L_1 on X.

Step 3: Indeed, the bundle $j^* L$ is trivial on the fibers of π, hence it is obtained as a pullback of a line bundle L_0 on $X \backslash D_1$. By the same argument, $\operatorname{Pic}(X) = \operatorname{Pic}(X \backslash D_1)$, hence L_0 is a restriction of a line bundle L_1 on X. Then $L\Big|_{M \backslash Z} = \pi^* L_1\Big|_{M \backslash Z}$. **Applying "Serre's condition S2" again, we obtain that $L = \pi^* L_1$.** ■

Multiplicity of the fibers

DEFINITION: Let $\pi : M \longrightarrow X$ be a proper holomorphic map, $x \in X$ a point, and $\mathfrak{m}_x$ it maximal ideal. Denote by $F_x := \pi^{-1}(x)$ the set-theoretic preimage of x, understood as a complex variety, and let $F_1, ..., F_k$ be its irreducible components. **(Scheme-theoretic) multiplicity of π in F_i** is the rank of $\mathcal{O}_M/\pi^*\mathfrak{m}_x$ in a general point of F_i.

REMARK: Let $\pi : M \longrightarrow X$ be a proper holomorphic map of complex manifolds, with $\dim X = 1$. Consider a general point z in a multiple fiber F_x. Assume that F_x has multiplicity k in z. **Then for an appropriate neighbourhood U_z of $z \in M$, U_z is locally homeomorphic to $U_F \times \Delta$, where Δ is a disk, U_F a neighbourhood of z in F_x, and $\pi(z_1, t) = t^k$. In particular, π restricted to Δ is a k-sheeted covering.** In these coordinates, the Clemens-Persson map acts on U_z by contracting Δ to the origin.

Multiple fibers

DEFINITION: Let $\pi : M \longrightarrow X$ be a proper holomorphic map, $x \in X$ a point, $F_x := \pi^{-1}(x)$ and F_i its irreducible component, with (scheme-theoretical) multiplicity μ_i. Denote the greatest common divisor of μ_i by μ. A fiber is **multiple** if $\mu > 1$. A fiber F_x is **reduced** if $\mu_i = 1$ for all i. A fiber F_x is **has a reduced component** if $\mu_i = 1$ for at least one i.

CONJECTURE: Let $\pi : M \longrightarrow X$ be a flat, proper holomorphic map, and $x \in X$, with general fiber a complex torus. Consider the Clemens-Persson retraction map $\Phi : \pi^{-1}(U) \longrightarrow F_x$. Let F be a smooth fiber of π. Assume that F_x is non-multiple. **Then $\Phi : F \longrightarrow F_x$ is surjective on fundamental groups.**

Theorem 2: This conjecture is true if F_x has a reduced component.
Proof: Next slide. ■

COROLLARY: Let M be a hyperkähler manifold and $\pi : M \longrightarrow X$ a Lagrangian fibration. Assume that for a general point x of the discriminant, F_x has a reduced component. **Then $\pi^* H$ is primitive for any primitive line bundle H on X.**
Proof: By Theorem 1, for primitivity of $\pi^* H$ it suffices to check that $\Phi : F \longrightarrow F_x$ is surjective on fundamental groups for x a general point in the discriminant. By Theorem 2, it is surjective for such x. ■

Surjectivity for a fiber which has a reduced component

Theorem 2: Let $\pi : M \longrightarrow X$ be a proper holomorphic map, and $x \in X$, with general fiber a complex torus. Consider the Clemens-Persson retraction map $\Phi : \pi^{-1}(U) \longrightarrow F_x$. Let F be a smooth fiber of π. Assume that F_x has a reduced component. **Then $\Phi : F \longrightarrow F_x$ is surjective on fundamental groups.**

Proof: Let $F_1 \subset F_x$ be a component which has multiplicity 1. Consider a loop γ in F_x which starts and ends in $y \in F_1$. The map $\Phi : \pi^{-1}(U) \longrightarrow F_x$ restricted to F is a ramified covering, which is bijective over a general point of F_1. Therefore, $\Psi^{-1}(\gamma) \cap F$ is a path in F which starts and ends in $\Psi^{-1}(y) \cap F$. Then Φ applied to this loop gives γ, and $\Phi : F \longrightarrow F_x$ is surjective on fundamental groups. ∎